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A regularity theorem for stationary varifolds and an existence theorem for mean curvature flow.

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To Ben and Jo,
this is all *their* fault.

Abstract

The goal of this thesis is to examine two variational problems in geometry arising from the study of the area functional. The first half is dedicated to studying the boundary regularity of critical points of the area functional, while the second half focusses on short time existence of smooth solutions to the L^2 gradient descent of the same functional.

We first study regularity of stationary integral n-varifolds that are L^2-close to a pair of planes intersecting along an $(n-1)$-dimensional subspace. We show that provided such a varifold V satisfies suitable mass bounds, the aforementioned L^2 distance is sufficiently small, and V satisfies certain structural assumptions on the singular set; then V consists of four smooth sheets meeting along a $C^{1,\alpha}$ curve. This immediately implies a corresponding boundary regularity result for subspace boundaries by Allard's reflection principle.

We also study short time existence of Lagrangian mean curvature flow from a non-smooth initial condition. In particular we show that for any compact Lagrangian $L \subset \mathbb{C}^n$ with a finite number of singularities, each asymptotic to a pair of non area minimising, transversally intersecting Lagrangian planes, there is a smooth Lagrangian mean curvature flow existing for some positive time, that attains L as $t \searrow 0$ as varifolds, and smoothly locally away from the singularities.

We aim to give a thorough account of each problem, while highlighting areas of overlap in the approaches that point to wider applicability of these methods to problems in geometric analysis and variational geometry in general.

Statement of originality

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution except as declared in the Preface and specified in the text.

Chapter 1 consists of motivation and a literature review. Chapter 2 contains background material on geometric measure theory. Chapter 3 contains regularity theorems for stationary varifolds that are the result of my own work. Chapter 4 contains background material on mean curvature flow and some complex geometry. Chapter 5 contains the proof of a short time existence theorem for Lagrangian mean curvature flow. This is the result of collaboration with Kim Moore and has appeared in [5]. The work done in collaboration is entirely contained in Section 5.6, the remainder of the chapter is the result of my own work.

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Contents

"Would it save you a lot of time if I just gave up and went mad now?"

Arthur Dent
The Hitchhiker's Guide to the Galaxy

Chapter 1

Introduction

In this thesis we study two problems in geometric analysis that share a common starting point; the area functional. While the problems themselves are quite different, one being elliptic, the other parabolic, they have many things in common, and often techniques that prove fruitful in the study of one problem turn out to have analogues in the other.

1.1 Geometric background

We begin by introducing the area functional. Consider an open set $U \subset \mathbb{R}^{n+k}$ and a closed (that is, compact and without boundary) C^1 manifold $M \subset U$. The area functional is simply the n-dimensional Hausdorff measure of M, denoted $\mathcal{H}^n(M)$ (see [51]). We will assume in this section that $\mathcal{H}^n(M) < \infty$. A natural question is to investigate how the area changes when we deform M. Suppose that we have a compactly supported, continuously differentiable vector field $X \in C^1_c(U; \mathbb{R}^{n+k})$. Given such an X, we define the one parameter family of maps

$$\varphi_t \colon U \to \mathbb{R}^{n+k}, \;\; x \mapsto x + tX(x),$$

which are bijective onto U if $|t|$ is sufficiently small. Thus, for $\varepsilon > 0$ small enough, the family $M_t := \varphi_t(M)$ for $t \in (-\varepsilon, \varepsilon)$ consists of C^1 closed submanifolds of U with finite $\mathcal{H}^n$-measure. To see how the area of M_t changes as t varies, we can compute the derivative of $\mathcal{H}^n(M_t)$ at $t = 0$ explicitly in terms of X as follows (see [51] for details).

Lemma 1.1.1 (First variation formula)**.** *Suppose that M is a closed C^1 submanifold of some open set $U \subset \mathbb{R}^{n+k}$ with finite $\mathcal{H}^n$-measure and that $X \in C^1_c(U;\mathbb{R}^{n+k})$, then with M_t defined as before we have*

$$\frac{d}{dt}\mathcal{H}^n(M_t)\bigg|_{t=0} = \int_M \operatorname{div}_M X \mathrm{d}\mathcal{H}^n, \tag{1.1.1}$$

where div_M *is the tangential divergence defined as* $\operatorname{div}_M X(x) := \sum_{i=1}^n \tau_i \cdot D_{\tau_i} X(x)$*, for $\tau_1, \ldots, \tau_n$ any orthonormal basis for T_xM and D_{τ_i} the directional derivative in the direction τ_i.*

Remark 1.1.2. *Notice that the fact that $\mathcal{H}^n(M_t)$ is finite for each M_t means this derivative is well-defined, with differentiability following from the fact that X is C^1. It is possible to still make sense of the above for M non-compact if we assume that M has locally finite $\mathcal{H}^n$-measure.*

If M is at least C^2, then one can show that for any vector field X, we have the pointwise identity

$$\operatorname{div}_M X^\perp(x) = -X^\perp(x) \cdot \vec{H}(x) = -X(x) \cdot \vec{H}(x), \tag{1.1.2}$$

where $\vec{H}(x)$ is the mean curvature vector at x and $(\cdot)^\perp$ denotes the projection to $(T_xM)^\perp$. Moreover by the divergence theorem, since we have that $\partial M \cap U = \emptyset$, it follows

$$\int_M \operatorname{div}_M X^T(x) \mathrm{d}\mathcal{H}^n = 0,$$

where $(\cdot)^T$ denotes the projection to T_xM. Hence we conclude, by combining Lemma 1.1.1, equation (1.1.2) and the divergence theorem that

$$\frac{d}{dt}\mathcal{H}^n(M_t)\bigg|_{t=0} = \int_M \operatorname{div}_M X \mathrm{d}\mathcal{H}^n = -\int_M X \cdot \vec{H} \mathrm{d}\mathcal{H}^n. \tag{1.1.3}$$

Given (1.1.3) we can consider both critical points and gradient flows for the area functional. Indeed we see that if $\vec{H} \equiv 0$ on M, then M is a critical point for area in the space of n-dimensional submanifolds of U. This observation forms the basis of the definition of a minimal surface, which we will introduce in more detail in Section 1.2. On the other hand, from the same formula we see that the most efficient way to reduce area would be to allow each point to move with velocity equal to the mean curvature vector. We call such a motion a mean curvature flow, and give a more detailed overview in section 1.3.

1.2 Minimal surfaces

As observed in Section 1.1, the first variation formula (1.1.3) implies that a C^2 submanifold, $M \subset \mathbb{R}^{n+k}$, with $\vec{H} \equiv 0$ is necessarily a critical point for the area functional.

Definition 1.2.1. *We say that a C^2 submanifold $M \subset \mathbb{R}^{n+k}$ with $\vec{H} \equiv 0$ is* minimal*, or a* minimal surface*.*

The study of minimal surfaces dates back to the beginnings of the calculus of variations and the work of Euler and Lagrange. A typical problem is the following: given an $(n-1)$-dimensional boundary in $\mathbb{R}^{n+k}$, find the n-dimensional surface of least area with that boundary. In the case $n = 2$, $k = 1$, this is known as Plateau's problem, named for Joseph Plateau, who experimented with soap films spanning wire frames in the late 19th century, and derived laws governing their structure and regularity [47]. The first general solutions to Plateau's problem were constructed simultaneously by Jesse Douglas [14] and Tibor Radó [48], for which the former won the Fields medal.

Of course to answer such questions one must first decide what we even mean by 'surface', and indeed what sort of assumptions we wish to make on the regularity of the boundary. Early investigations of Plateau's problem, including the work of Douglas and Radó, generally defined surfaces to be mappings of a disk. While this approach has its advantages, there are also some serious limitations. In particular, viewing a surface as a mapping of a disk places an a priori restriction both on the types of singularities that can arise, as well as the topological complexity. Another problem is that such a class lacks good compactness properties when endowed with any natural topology. Such properties are desirable when addressing questions of existence, or when conducting a blow-up analysis. For these reasons it is desirable to work in a larger class that generalises C^1 submanifolds with locally finite $\mathcal{H}^n$-measure.

There are multiple possible choices for such a generalised class, but any class we choose should have a few basic properties. Firstly, one needs a notion of area that extends that of $\mathcal{H}^n$-measure restricted to the surface; secondly, it should be possible to extend the variational notion of minimality given in (1.1.1); and finally it would be desirable if area were continuous with respect to the topology of this space, or at least continuous on the subset of critical points.

A natural notion of surface is that of a countably n-rectifiable set, which we will define rigorously in Chapter 2. Informally speaking, a countably n-rectifiable set can be thought of as a $\mathcal{H}^n$-measurable set M with locally finite $\mathcal{H}^n$-measure that, away from a set of $\mathcal{H}^n$-measure zero, is a countable union of embedded C^1 n-dimensional submanifolds. An important equivalent definition is that M possesses a well-defined measure-theoretic notion of tangent plane at almost every point. Given this it is possible to define notions of tangential derivatives and hence one can give meaning to the right hand side of (1.1.1) for M merely n-rectifiable.

Though this class is promising, it doesn't possess all the properties we require. Indeed one needs to expand the class further to the space of integer multiplicity rectifiable n-varifolds, hereafter referred to simply as rectifiable n-varifolds or varifolds (see Chapter 2). This space was originally introduced by Almgren in [3], before later being streamlined by Allard in [1], see also Simon [51]. It consists of pairs of countably n-rectifiable sets M and functions $\theta\colon M \to \mathbb{N}$, called the 'multiplicity'. The area, or indeed 'mass', is defined to be $\mathcal{H}^n \llcorner \theta$; that is 'sheets' of M are counted with multiplicity. Allowing this is crucial if we hope to have continuity of area, as evidenced by the example of a sequence of two planes coming together in the limit. Notice in particular that any C^1 submanifold with locally finite $\mathcal{H}^n$-measure can clearly be viewed as a rectifiable n-varifold with unit multiplicity everywhere.

The space of rectifiable n-varifolds has been successfully used to answer deep geometric questions, for example the existence of minimal surfaces of specific dimension in an arbitrary compact Riemannian manifold, which was established through work of Almgren [3], Pitts [46] and Schoen-Simon [50].

One can draw an analogy between using the class of rectifiable n-varifolds to study problems in geometry, and using Sobolev spaces to study partial differential equations; in both cases one has to sacrifice a priori regularity to gain compactness. Thus while questions of existence generally become easier, the real challenge is in the regularity theory. We hope that a member of this class that solves some partial differential equation or minimisation problem, will be much more regular than a typical member of the class.

1.2.1 Interior regularity of stationary varifolds

We defer a detailed discussion of the technicalities to Chapter 2, but we remark that it is possible to derive an analogue of (1.1.1) for integral varifolds. In particular we can deform a varifold along a vector field and compute the derivative of the area. Critical points for area are precisely those varifolds for which this derivative is zero for any valid choice of vector field.

Definition 1.2.2. *We call a varifold V that is critical for area in the aforementioned variational sense* stationary*.*

Since we want to investigate the regularity of stationary varifolds, we make the following definition.

Definition 1.2.3. *Given a varifold V, we define the* regular set *of V, denoted* $\text{reg}V$*, to be the set of all points $x \in V$ such that there is an open neighbourhood of x in which V is a smooth n-dimensional submanifold. We define the* singular set*, denoted* $\text{sing}V$*, to be the set of all points $x \in V$ such that $x \notin \text{reg}V$.*

Remark 1.2.4. *It is a slight abuse of notation to say $x \in V$, but we do so in order to not get mired in the technicalities of the definition of a varifold here. The above definition is made more precise in Chapter 2.*

There is little known in general about the regularity of stationary varifolds. For example, it is in fact still an open question whether $\mathcal{H}^n(\text{sing}V) = 0$ for $n \geq 2$. One of the few known results is due to Allard [1], who in his seminal paper was able to prove the following

Theorem 1.2.5 (Allard)**.** *The regular set of any stationary varifold is open and dense.*

In fact Allard was able to prove much more than this, but a precise statement requires the introduction of much additional terminology, which we defer to Chapter 2. Instead we simply remark that the stationarity assumption itself can be relaxed to assuming that the mean curvature (or rather a generalisation thereof) is in L^p for some $p > n$.

Under additional assumptions on the varifold, considerably more is known.

- If $k = 1$ and V is area minimising, then through combined work of DeGiorgi [12], Federer-Fleming [20], Simons [54], and Federer [19] it has been shown that the singular set is codimension 7, i.e. $\dim_{\mathcal{H}}\mathrm{sing}V \leq n - 7$. Here, to make sense of what it means to be area minimising, V must correspond to an area minimising rectifiable current. This is an alternative generalisation of C^1 submanifolds, which we do not define here, that comes equipped with a notion of orientation and boundary. Area minimising means that any compact piece of the surface does not have greater area than any competitor piece with the same boundary.

- If $k \geq 2$ and V is area minimising, then work of Almgren [4] shows that $\dim_{\mathcal{H}}\mathrm{sing}V \leq n-2$. In this case and also the codimension 1 case, there are well known examples to show that the dimension bounds are sharp.

- If $k = 1$ and V is stationary; stable, which is to say that the second variation is non-negative; and V has no 'classical singularities', i.e. no singularities for which one can find a neighbourhood in which V consists of 3 or more $C^{1,\alpha}$ sheets meeting along a common boundary; then work of Wickramasekera [64] implies $\dim_{\mathcal{H}}\mathrm{sing}V \leq n - 7$. This fully generalises the $k = 1$ area minimising case, since area minimising V are necessarily stationary, stable, and do not have classical singularities.

1.2.2 Boundary regularity of stationary varifolds

The above results are all statements about the interior. When it comes to boundary regularity, much less is known. Allard [2] proved a boundary analogue of his interior regularity result, Theorem 1.2.5, which implies the following.

Theorem 1.2.6. *Let B be a $C^{1,1}$ curve in $B_1(0)$ with $0 \in B$. Suppose that V is a stationary varifold in $B_1(0) \setminus B$ which is asymptotic to a half plane H at 0 with $\partial H = T_0B$, in the sense that there is a sequence of rescalings of V that converges to H. Then in a neighbourhood U of 0, V consists of a smooth manifold M with $\partial M \cap U = B \cap U$.*

Remark 1.2.7. *As before, we note that it is possible to make a much more precise statement, but we defer this to Section 3.1.*

The requirement that B is $C^{1,1}$ has been relaxed to $C^{1,\alpha}$ for any $\alpha \in (0,1)$ by Bourni [7]. Furthermore, as is the case for Theorem 1.2.5, the requirement that V is stationary can instead be relaxed to the generalised mean curvature being in L^p for $p > n$.

Unlike the interior case, there has not been much more than this proved even under additional assumptions. One of the few examples is the following.

- If $k = 1$ and V corresponds to an area minimising rectifiable current T and ∂T is connected, oriented and embedded, then Hardt-Simon [26] showed in a neighbourhood of ∂T, T is a $C^{1,\alpha}$ connected, embedded submanifold.

One of the main difficulties lies in the fact that the varifold is only assumed stationary in the complement of the boundary curve. Consequently, the estimates that prove so successful in proving interior regularity theorems do not in general hold at the boundary.

Despite the difficulties, in light of Theorem 1.2.6 it is natural to ask the following question: "if at a boundary point, a stationary varifold V is asymptotic to a pair of half-planes meeting along a common boundary, what can we say about the regularity of V in a neighbourhood of that point?". In particular we want to understand if Allard's ideas in the proof of Theorem 1.2.6 could be adapted to the case of two half-planes, and if we can obtain the analogous conclusion, namely that in a small neighbourhood of the point on the boundary the varifold consists of two smooth manifolds meeting along the boundary.

One of the key tools in Allard's proof was the following reflection principle.

Lemma 1.2.8 (Reflection Principle)**.** *Let P be an $(n-1)$-dimensional subspace passing through the origin, and let p and $p_\perp$ denote the orthogonal projections to P and $P^\perp$ respectively. If V is stationary in $B_1(0) \setminus P$, then $\hat{V} := V + \tilde{V}$ is stationary in $B_1(0)$, where $\tilde{V}$ is the reflection of V, i.e. the 'image' of V under the map $\vartheta\colon x \mapsto p(x) - p_\perp(x)$.*

In the case of a subspace boundary, the reflection principle effectively transforms boundary points into interior points, as the reflected varifold is stationary across the boundary. Allard was able to use this observation successfully to allow him to use his interior regularity theorem in addressing the boundary regularity question. If we wish to do the same thing in the case of a varifold asymptotic to two half-planes, we need a corresponding interior theorem. In particular we need to show that a varifold close in mass and in L^2 to a pair of planes intersecting along an $(n-1)$-dimensional subspace, consists in the interior of four smooth submanifolds meeting along a common boundary.

Unfortunately, as was remarked already, assuming only stationarity, not much is known beyond Allard's interior theorem (Theorem 1.2.5). Even worse, the theorem we need, as stated above, is false, and there are simple counter examples (see Section 3.1). Instead we must make more restrictive assumptions. Natural additional assumptions, for example that the surface be stable, or area minimising, do not seem to behave well with the reflection principle, so it is unclear how to proceed in this case. Instead we make an a priori assumption about the structure of the singular set, and show that under these assumptions, we can prove the aforementioned interior theorem, which we may state informally as follows.

Theorem 1.2.9 (Main regularity theorem)**.** *If V is a varifold, which is stationary in $B_1(0)$ and is sufficiently close in mass and in L^2 to a pair of planes intersecting along an $(n-1)$-dimensional subspace, and if the singular set of V satisfies certain structural assumptions, then in a neighbourhood U of the origin, V consists of four smooth submanifolds with a common $C^{1,\alpha}$ boundary in U.*

Using the reflection principle we can immediately deduce a corresponding boundary regularity result, in the case that the boundary is an $(n-1)$-dimensional subspace. Whether or not the interior regularity theorem can be used to prove corresponding boundary regularity results for more general boundaries remains open.

In Chapter 2 we introduce in detail the concept of stationary varifolds, and state related results and definitions. Chapter 3 is dedicated to proving the above regularity theorem and corollary. We will also give a more detailed and technical motivation of the problem in Section 3.1.

1.3 Mean curvature flow

Returning to (1.1.3), we see that the most efficient way to decrease area would be to choose the vector field X to coincide with the mean curvature vector of M at each point. This motivates the definition of mean curvature flow, which is the gradient descent for the area.

Definition 1.3.1. *A one parameter family of evolving surfaces M_t is a* mean curvature flow *if the normal velocity at each point is equal to the mean curvature vector.*

Remark 1.3.2. *Strictly speaking this is not a gradient descent in the classical sense, as there is no fixed L^2 structure, indeed the surface measure evolves with the moving hypersurfaces. It does however behave much like one would expect a classical gradient descent to behave.*

The mean curvature flow first appeared in the work of material scientists studying annealing metals, bubble growth, and other physical phenomena where systems evolve so as to minimise their surface area. In particular Mullins [43], in his investigation of moving grain boundaries, may have been the first to write down the mean curvature flow equation, and also was able to find certain self-similar solutions. Brakke [8] later independently defined a general measure-theoretic notion of the flow, see Section 4.4 for more details on his construction.

In both cases the mean curvature flow can be shown to be equivalent to a quasilinear second order parabolic partial differential equation being satisfied on the surface. Solutions of the mean curvature flow exhibit many properties that one would expect of solutions to such an equation; for example, short time existence and uniqueness for a fairly general class of initial conditions.

In some cases, the behaviour of solutions is well understood. In his seminal paper, Huisken [28] considered a classical parametric formulation of the mean curvature flow akin to that of Mullins, and showed that closed convex hypersurfaces of dimension $n \geq 2$ contract to a 'round point'. That is to say that if the evolution is rescaled so as to keep enclosed volume constant, then the rescaled hypersurfaces converge to a round sphere. In the one-dimensional case, combined work of Gage-Hamilton [21] and Grayson [23] showed that any embedded closed curve in the plane shrinks to a round point under mean curvature flow (also called curve shortening flow in this case).

Huisken's result is no longer true if one drops the assumption of convexity. Indeed the standard example is that of a 'dumbbell', two large spheres joined together by a narrow cylinder. The spheres being large, have small curvature, and so only contract inwards slightly in a short period of time. The cylinder on the other hand, being very narrow, will contract inwards much more quickly, pinching off at a point. Thus it is certainly possible for flows to develop singularities without vanishing completely.

In general, the evolution of a closed submanifold of Euclidean space will develop a singularity in finite time. Moreover one can easily check that if a singularity develops at a time $T \in (0, \infty)$, then the curvature must become unbounded as $t \nearrow T$, so it is not possible to extend the flow classically. We want to understand the behaviour of the solution as we approach the singular time, in the hope that it might nevertheless be possible to continue the flow in a controlled way.

There are different ways we might try to extend the flow. One possibility is to adopt a weaker notion of solution that allows for the presence of singularities. Various weak formulations of the mean curvature flow have been introduced, the earliest being the aforementioned measure theoretic solutions of Brakke [8], with later refinements due to Ilmanen [32]. Chen-Giga-Goto [11] and Evans-Spruck [17] also introduced a level set formulation, based on the theory of viscosity solutions of partial differential equations.

An alternative approach, inspired by work of Hamilton and Perelman on the Ricci flow, is to perform surgery on the surfaces. Here one cuts out regions of high curvature before a singularity can develop, and replaces them with something more regular. One can then continue flowing, and study the evolution of the resulting pieces. This approach has the advantage of being able to keep track of changes in topology, and apart from the surgery times themselves, the flows remain smooth. There are drawbacks however, for example there is no canonical way to perform a surgery, so instead a choice has to be made resulting in non-uniqueness. Moreover the flow with surgeries is not really a solution to the original problem. Though it can be used to deduce interesting facts about geometry or topology, from the perspective of analysing the underlying partial differential equations it is less interesting. Surgery procedures have been successfully carried out for mean curvature flow of hypersurfaces M in $\mathbb{R}^{n+1}$ by Huisken-Sinestrari [30] for $n \geq 3$ if M is assumed 2-convex, i.e. that the sum of the smallest two principle curvatures is everywhere non-negative; and by Brendle-Huisken [9] if $n = 2$ and M is assumed mean convex, i.e. that the (scalar) mean curvature is everywhere non-negative. Notice in particular that in the $n = 2$ case, the notions of mean convexity and 2-convexity coincide.

One final possibility for extending a flow is to take a weak limit of the flow at the first singular time, and then try to prove short time existence of a smooth solution that attains the singular limit as its initial condition in some suitable sense. This is the approach we take in Chapter 5. In particular, we are motivated by a problem from complex geometry, which can be rephrased as a question of the existence of minimal surfaces in certain homology classes. The use of mean curvature flow has been suggested as a potential means of solving this problem, but singularities can be shown to develop for generic initial conditions, in the sense that given any initial condition, one can find another initial condition in the same class that develops a finite time singularity. Consequently, if the mean curvature flow is to be used to construct a minimal surface in this class, one needs to be able to continue the flow past singularities. We will show that for certain types of singularities that develop under the flow, it is possible to continue the flow past the singularity, with every time slice consisting of smooth submanifolds except at the singular time.

Chapter 4 contains the definition of mean curvature flow along with some basic results. In Chapter 5 we prove short time existence of smooth flows originating from certain kinds of singular initial condition.

1.4 Notation

We collect here some of the basic notation used throughout the thesis. Notation specific to later chapters will be defined as needed.

- n and k will denote positive integers. We work in $\mathbb{R}^{n+k}$, n will usually be reserved for the dimension of the object of study, while k will denote the codimension.

- Given $x \in \mathbb{R}^{n+k}$ and $\rho > 0$ we denote by $B_\rho(x)$ the open ball of radius ρ centred at x, that is $B_\rho(x) = \{y \in \mathbb{R}^{n+k} \mid |x - y| < \rho\}$. In the case $x = 0$ we typically abbreviate this to B_ρ.

- Given $x \in \mathbb{R}^m$, we denote by $B^m_\rho(x)$ the m-dimensional ball of radius ρ centred at x.

- For $s \geq 0$ we denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure on $\mathbb{R}^{n+k}$. For $m \in \mathbb{N} \cup \{0\}$ we let ω_m be the volume of the m-dimensional unit ball, i.e. $\omega_m = \mathcal{H}^m(B^m_1(0))$. Here we interpret $B^m_1(0)$ as a subset of $\mathbb{R}^{n+k}$ by identifying with $B^m_1(0) \times \{0\}^{n+k-m}$.

- For A, $B \subset \mathbb{R}^{n+k}$ we denote by $\mathrm{dist}_{\mathcal{H}}(A, B)$ the Hausdorff distance between A and B.

- For $A \subset \mathbb{R}^{n+k}$ we denote by $\dim_{\mathcal{H}} A$ the Hausdorff dimension of A, which is defined as the infimum over all $s \geq 0$ for which $\mathcal{H}^s(A) = 0$.

- Given a measure μ on $\mathbb{R}^{n+k}$ and a μ-measurable subset $A \subset \mathbb{R}^{n+k}$ we denote by $\mu \llcorner A$ the restriction of μ to A, i.e. the measure defined by

$$\mu \llcorner A(B) := \mu(A \cap B), \qquad B \subset \mathbb{R}^{n+k} \ \mu\text{-measureable.}$$

  Given a μ-measurable function $f\colon \mathbb{R}^{n+k} \to [0, \infty)$, we define

$$\mu \llcorner f(B) := \int_B f \mathrm{d}\mu, \qquad B \subset \mathbb{R}^{n+k} \ \mu\text{-measureable.}$$

  In particular, we have $\mu \llcorner A = \mu \llcorner \mathbb{1}_A$.

Chapter 2

Geometric measure theory

In this chapter we introduce concepts from geometric measure theory that will be required later. Of particular interest to us are so-called stationary varifolds, informally introduced in Section 1.2, which are a measure-theoretic notion of minimal surface that allow for the presence of singularities. Much of the material in this chapter is well known, and unless otherwise specified a good reference is [51].

2.1 Rectifiable sets and rectifiable varifolds

The generalised notion of submanifold we use is that of an integer multiplicity n-rectifiable varifold. As stated in Section 1.2 these can be thought of as 'surfaces with multiplicity', where 'surface' is interpreted as a countably n-rectifiable set. We begin by giving the formal definition of a countably n-rectifiable set, as well as some basic results concerning their structure.

2.1.1 Countably n-rectifiable sets

Definition 2.1.1 (Countably n-rectifiable set)**.** *We say that a set $M \subset \mathbb{R}^{n+k}$ is* countably n-rectifiable *if*

$$M \subset M_0 \cup \bigcup_{j=1}^{\infty} F_j(\mathbb{R}^n)$$

where $M_0 \subset \mathbb{R}^{n+k}$ satisfies $\mathcal{H}^n(M_0) = 0$, and each $F_j \colon \mathbb{R}^n \to \mathbb{R}^{n+k}$ is Lipschitz.

Remark 2.1.2. *By the extension theorem for Lipschitz functions, we have that*

M is countably n-rectifiable if and only if

$$M = M_0 \cup \bigcup_{j=1}^{\infty} F_j(A_j)$$

where $\mathcal{H}^n(M_0) = 0$ and $F_j \colon A_j \to \mathbb{R}^{n+k}$ is Lipschitz with $A_j \subset \mathbb{R}^n$ for each j.

The following important characterisation of countably n-rectifiable sets is sometimes even taken as the definition. It serves as an informal justification for why we might expect countably n-rectifiable sets to be good models for the limits of sequences of C^1 submanifolds.

Theorem 2.1.3. *A subset $M \subset \mathbb{R}^{n+k}$ is countably n-rectifiable if and only if*

$$M \subset N_0 \cup \bigcup_{j=1}^{\infty} N_j,$$

where $N_0 \subset \mathbb{R}^{n+k}$ satisfies $\mathcal{H}^n(N_0) = 0$, and N_j is an n-dimensional embedded C^1 submanifold of $\mathbb{R}^{n+k}$ for each $j \geq 1$.

In order to study the geometry of critical points of the area functional, it is desirable to have some notion of a tangent space so that tangential derivatives can be defined. We will define so-called approximate tangent spaces to be linear subspaces arising as limits of rescalings of a set M in some appropriate topology. We first introduce the following rescaling function.

Definition 2.1.4. *We define $\eta_{x,\rho} \colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ to be the function which first translates x to the origin and then rescales by a factor ρ^{-1}, that is to say*

$$\eta_{x,\rho}(y) := \frac{y - x}{\rho}.$$

With this in hand we define the approximate tangent space of a measurable subset $M \subset \mathbb{R}^{n+k}$.

Definition 2.1.5. *Suppose that $M \subset \mathbb{R}^{n+k}$ is $\mathcal{H}^n$-measurable and $\theta \colon M \to (0, \infty)$ is locally $\mathcal{H}^n$-integrable. We say that the n-dimensional subspace $P \subset \mathbb{R}^{n+k}$ is the* approximate tangent space *for M at x with respect to θ if*

$$\lim_{\rho \searrow 0} \int_{\eta_{x,\rho}(M)} f(y)\theta(x + \rho y) \mathrm{d}\mathcal{H}^n(y) = \theta(x) \int_P f(y) \mathrm{d}\mathcal{H}^n(y),$$

for every compactly supported continuous function $f \in C_c(\mathbb{R}^{n+k})$.

Remark 2.1.6. *If* M *has locally finite* $\mathcal{H}^n$*-measure, so that* $\mathcal{H}^n(M \cap K) < \infty$ *for each compact* $K \subset \mathbb{R}^{n+k}$*, then we can take* $\theta \equiv 1$*, and the definition is equivalent to saying that the Radon measures* $\mathcal{H}^n \llcorner \eta_{x,\rho}(M)$ *converge in the usual weak* sense to* $\mathcal{H}^n \llcorner P$ *as* $\rho \searrow 0$.

Notice also that if $M \subset \mathbb{R}^{n+k}$ *is* $\mathcal{H}^n$*-measurable, and* $\theta \colon M \to (0, \infty)$ *is locally* $\mathcal{H}^n$*-integrable, then the set* $M_\delta := \{x \in M \mid \theta(x) \geq \delta\}$ *has locally finite* $\mathcal{H}^n$*-measure. We can then show that for* $\mathcal{H}^n$*-almost every* $x \in M_\delta$*, the approximate tangent space to* M *with respect to* θ *coincides with the approximate tangent space to* M_δ *with respect to* $\theta \equiv 1$*. Hence we see that for any two choices of function,* θ *and* $\tilde{\theta}$*, the approximate tangent spaces to* M *with respect to* θ *and* $\tilde{\theta}$ *coincide* $\mathcal{H}^n$*-almost everywhere. In light of this we denote the approximate tangent space to* M *at* x *by* T_xM *wherever it exists.*

The assumptions we make on M in the above definition in no way guarantee the existence of approximate tangent spaces at any given point. It turns out there is an important characterisation of countably n-rectifiable sets in terms of the existence of approximate tangent spaces.

Theorem 2.1.7. *Suppose that* $M \subset \mathbb{R}^{n+k}$ *is* $\mathcal{H}^n$*-measurable. Then* M *is countably* n*-rectifiable if and only if there is* $\theta \colon M \to (0, \infty)$*, a locally* $\mathcal{H}^n$*-integrable function with respect to which there exists an approximate tangent space to* M *at* $\mathcal{H}^n$*-almost every* $x \in M$.

Remark 2.1.8. *The proof uses the fact that if* M *is countably* n*-rectifiable, then it can be written in the form* $M = M_0 \cup \bigcup_j M_j$ *where* $\mathcal{H}^n(M_0) = 0$ *and the* M_j *are pairwise disjoint and* $M_j \subset N_j$ *for each* j*, where* N_j *is an* n*-dimensional* C^1 *embedded submanifold. In this case, one can show that for* $\mathcal{H}^n$*-almost every* $x \in M_j$*, we have that* T_xM *exists and* $T_xM = T_xN_j$.

2.1.2 Tangential derivatives

Given a countably n-rectifiable set M and a point $x \in M$ at which T_xM exists, it is possible to define tangential notions of gradient, divergence and so on by projecting onto the tangent space in the appropriate way. For M countably n-rectifiable the approximate tangent space exists at almost every point, and hence these notions of tangential derivatives are well-defined as locally L^1 functions for instance. This allows us to make sense of what it means for a function to satisfy a partial differential equation in an integral sense on M.

Definition 2.1.9 (Tangential gradient)**.** *Given an n-dimensional subspace $S \subset \mathbb{R}^{n+k}$ and a C^1 function $f\colon \mathbb{R}^{n+k} \to \mathbb{R}$ we define the* gradient on S *to be*

$$\nabla^S f(x) := p_S(Df(x)),$$

where $D(\cdot)$ denotes the usual gradient on $\mathbb{R}^{n+k}$ and p_S denotes the orthogonal projection onto S. Given $M \subset \mathbb{R}^{n+k}$ countably n-rectifiable we define the gradient on M *by*

$$\nabla^M f(x) := \nabla^{T_xM} f(x) = p_{T_xM}(Df(x)),$$

for any x at which the approximate tangent space T_xM exists.

If M happened to be a smooth embedded submanifold of $\mathbb{R}^{n+k}$, then ∇^M would be well-defined at every point and coincide with the usual notion of gradient inherited from the ambient space $\mathbb{R}^{n+k}$.

We can define the divergence in a similar manner.

Definition 2.1.10 (Tangential divergence)**.** *Suppose that $S \subset \mathbb{R}^{n+k}$ is an n-dimensional subspace and that $X\colon \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is a C^1 vector field. We define the* divergence on S *by*

$$\mathrm{div}_S X(x) := \mathrm{tr}_S(DX(x)) = \sum_{i,j=1}^{n+k} p^{ij} D_i X_j(x) = \sum_{i=1}^{n} \tau_i \cdot D_{\tau_i} X,$$

where (p^{ij}) denotes the matrix of the orthogonal projection to S, $\{\tau_1, \ldots, \tau_n\}$ is an orthonormal basis for S, and $D_{\tau_i}(\cdot)$ is the directional derivative in the direction τ_i. As before, given M countably n-rectifiable, we define the divergence on M *at any x where T_xM exists by*

$$\mathrm{div}_M X(x) := \mathrm{div}_{T_xM} X(x).$$

Finally we define the differential as follows.

Definition 2.1.11 (Differential)**.** *Given $f\colon \mathbb{R}^{n+k} \to \mathbb{R}^N$, where $N \geq 1$, the* differential *at a point* $x \in \mathbb{R}^{n+k}$ *is the linear map*

$$df_x \colon \mathbb{R}^{n+k} \to \mathbb{R}^N, \;\; \tau \mapsto D_\tau f(x),$$

where D_τ *is the directional derivative in the direction* τ*. In particular, if* M *is countably* n*-rectifiable and* $x \in M$ *is a point where* T_xM *exists, then we define the differential on* M *at* x *to be the restriction* $d^M f_x := df_x|_{T_xM}$*.*

2.1.3 Rectifiable n-varifolds

We can now define a rectifiable n-varifold to be a countably n-rectifiable set together with a multiplicity function.

Definition 2.1.12 (Rectifiable n-varifold)**.** *A* rectifiable n-varifold $V = \mathbf{v}(M, \theta)$ *is the equivalence class of the pair* (M, θ)*, where* M *is* $\mathcal{H}^n$*-measurable and countably* n*-rectifiable and* $\theta \colon M \to (0, \infty)$ *is locally integrable, under the equivalence relation* $(M, \theta) \sim (M', \theta')$ *if and only if* $\mathcal{H}^n((M \setminus M') \cup (M' \setminus M)) = 0$ *and* $\theta = \theta'$ $\mathcal{H}^n$*-almost everywhere on* $M \cap M'$*. We say* V *is an* integer multiplicity rectifiable n-varifold*, or more briefly an* integral n-varifold*, if* θ *takes values in the positive integers.*

Notice that any C^1 submanifold of $\mathbb{R}^{n+k}$ automatically induces an integral n-varifold of the form $|M| = \mathbf{v}(M, \mathbb{1}_M)$. We will use the notation $|\cdot|$ to denote the multiplicity one varifold corresponding to a smooth submanifold.

Definition 2.1.13 (Weight measure)**.** *For any rectifiable* n*-varifold* $V = \mathbf{v}(M, \theta)$ *we define the* weight measure

$$\|V\| := \mathcal{H}^n \llcorner \theta,$$

where we understand $\theta \equiv 0$ *on the complement of* M*. In other words, for any* $\mathcal{H}^n$*-measurable subset* $A \subset \mathbb{R}^{n+k}$ *we have*

$$\|V\|(A) := \int_A \theta \mathrm{d}\mathcal{H}^n = \int_{A \cap M} \theta \mathrm{d}\mathcal{H}^n.$$

Notice that since θ *is assumed to be locally integrable, it follows that* $\|V\|$ *is a Radon measure on* $\mathbb{R}^{n+k}$*.*

Given an integral n-varifold $V = \mathbf{v}(M, \theta)$, we allow V to inherit the notions of tangential derivative of Section 2.1.2 corresponding to the underlying countably n-rectifiable set M (which we can take to be $\mathrm{spt}\|V\|$). Thus, for example, the tangential gradient on V is denoted ∇^M. We will often use this notation without explicitly stating the relationship between M and V unless there is the possibility for confusion.

2.2 Stationarity and compactness

One of our primary motivations for introducing the concept of a varifold is that we would like to be able to use compactness theorems to do blow-up analysis of minimal surfaces. However given a sequence of integral n-varifolds V^j, and using only compactness theorems for Radon measures applied to a sequence $\|V^j\|$, we can't say much more about the limit beyond the fact that it's a Radon measure. It is unclear under what circumstances it actually corresponds to an integral n-varifold for example. In order to pass geometric information to the limit we must taken an even broader view, which is why we introduce the notion of a general n-varifold.

2.2.1 General n-varifolds

Like rectifiable n-varifolds, general n-varifolds are still Radon measures, but without any restriction on the geometry of the support. This allows for very wild behaviour, and also means that approximate tangent spaces will not exist in general. Instead we build tangent space information into the measure itself by considering measures over the Grassmann bundle.

Definition 2.2.1. *We denote by $G(n+k, n)$ the Grassmannian, that is to say the space of all n-dimensional linear subspaces of $\mathbb{R}^{n+k}$. Given a set $A \subset \mathbb{R}^{n+k}$ we denote by $G_n(A)$ the* Grassmann bundle over *A, that is*

$$G_n(A) = \{(x, S) \mid x \in A,\ \ S \in G(n+k, n)\} = A \times G(n+k, n).$$

Definition 2.2.2. *Given an open set $U \subset \mathbb{R}^{n+k}$, a* general n-varifold *V in U is a Radon measure on $G_n(U)$.*

Remark 2.2.3. *Any rectifiable n-varifold $V = \mathbf{v}(M, \theta)$ induces a corresponding general n-varifold via the formula*

$$V(A) := \|V\|(\pi(TM \cap A)),\ \ A \subset G_n(U)$$

where π is the projection onto the Euclidean factor of $G_n(U)$ and

$$TM := \{(x, T_xM) \mid x \in M,\ T_xM \ \text{exists at}\ x\}.$$

We endow the space of general n-varifolds with the weak* topology of Radon measures on $G_n(U)$. In particular we have the following definition.

Definition 2.2.4. *Suppose that V^j for $j \geq 1$ and V are general n-varifolds on some open set $U \subset \mathbb{R}^{n+k}$. We say that $V^j \to V$ as varifolds as $j \to \infty$ if*

$$\lim_{j\to\infty} \int_{G_n(U)} f(x, S)\mathrm{d}V^j(x, S) = \int_{G_n(U)} f(x, S)\mathrm{d}V(x, S),$$

for every $f \in C_c(G_n(U))$.

We also define the weight measure of a general n-varifold as follows.

Definition 2.2.5. *Given a general n-varifold V in $U \subset \mathbb{R}^{n+k}$, we define the* weight measure *as follows:*

$$\|V\|(A) := V(G_n(A)) = \int_{G_n(A)} \mathrm{d}V(x, S).$$

Notice in particular that if V is rectifiable then this definition coincides with the previous definition of weight measure for a rectifiable n-varifold.

Remark 2.2.6. *Varifold convergence implies convergence of the weight measures as Radon measures, and hence also that the supports of the weight measures converge locally in Hausdorff distance.*

2.2.2 First variation and stationarity

We wish to use varifolds as models for minimal surfaces, and so the first step is to develop an analogue of the first variation formula (1.1.1). To do so we need to be able to deform varifolds along a vector field, which requires the notion of an image varifold.

Definition 2.2.7. *Let V be a general n-varifold. Suppose that U, $W \subset \mathbb{R}^{n+k}$ are open, and $f\colon U \to W$ is C^1 and that f restricted to* $\mathrm{spt}\|V\| \cap U$ *is proper. Then we define the* image varifold $f_\# V$ *in W by*

$$f_\# V(A) := \int_{F^{-1}(A)} J_S f(x) \mathrm{d}V(x, S) \tag{2.2.1}$$

for any Borel set $A \subset G_n(W)$, and where the function $F\colon G_n^+(U) \to G_n(W)$ is defined $F\colon (x, S) \mapsto (f(x), \mathrm{d}f_x(S))$, and

$$\begin{aligned} J_S f(x) &= (\det((df_x|_S)^* \circ (df_x|_S)))^{1/2} \qquad \text{for all } (x, S) \in G_n(U), \\ G_n^+(U) &= \{(x, S) \in G_n(U) \mid J_S f(x) \neq 0\}, \end{aligned}$$

where $(df_x|_S)^*$ denotes the adjoint of $df_x|_S$.

Remark 2.2.8. *If $V = \mathbf{v}(M, \theta)$ is a rectifiable n-varifold, then*

$$f_\# V = \mathbf{v}(f(M), \overline{\theta}),$$

where

$$\overline{\theta}(x) = \sum_{y \in f^{-1}(x)} \theta(y).$$

Furthermore, if f is one-to-one then $\overline{\theta}(x) = \theta(f^{-1}(x))$. Notice that $\overline{\theta}$ is locally integrable by the area formula, and that in fact

$$\|f_\# V\|(W) = \int_W \mathrm{d}\|f_\# V\| = \int_U J_M f \mathrm{d}\|V\|,$$

where $J_M f$ is the Jacobian

$$J_M f = \sqrt{\det((d^M f_x)^* \circ d^M f_x)}.$$

Given this notion of an image varifold, we can now define the first variation of a general n-varifold.

Definition 2.2.9. *If V is a general n-varifold, then the* first variation, *which we denote δV, is a linear functional on $C^1_c(U; \mathbb{R}^{n+k})$ defined as follows. Given $X \in C^1_c(U; \mathbb{R}^{n+k})$ and $K \subset\subset U$ with* $\mathrm{spt}X \subset\subset K$, *we denote by φ_t the one-parameter family of diffeomorphisms $\varphi_t(x) := x + tX(x)$. Notice that for $|t|$*

sufficiently small, φ_t are one-to-one onto U. We set

$$\delta V(X) := \frac{d}{dt}\|\varphi_{t\#}V\|(K)\bigg|_{t=0}.$$

Remark 2.2.10. *By differentiating under the integral in* (2.2.1)*, one can show (see [51] for details)*

$$\delta V(X) = \int_{G_n(U)} \mathrm{div}_S X(x) \mathrm{d}V(x,S),$$

where div_S *is as defined in Section 2.1. Note in particular that if $V = \mathbf{v}(M,\theta)$ is rectifiable, then the first variation can be written*

$$\delta V(X) = \int_U \mathrm{div}_M X(x) \mathrm{d}\|V\|(x).$$

Remark 2.2.11. *We restrict to K in order that the derivative is well-defined, since $\varphi_{t\#}V$ is only guaranteed to have locally finite mass. Because* $\mathrm{spt}X \subset\subset K$*, we are discarding a part of V which remains fixed as t varies.*

We can now define what it means for V to be a critical point of the area functional.

Definition 2.2.12. *We say that V is* stationary *if $\delta V(X) = 0$ for every $X \in C^1_c(U;\mathbb{R}^{n+k})$.*

Remark 2.2.13. *Notice in particular, that if V corresponds to a classical submanifold, then V is stationary if and only if the corresponding submanifold is minimal.*

More generally we say that V has *locally bounded first variation* in U if for every compact subset $W \subset\subset U$ we have

$$|\delta V(X)| \leq C \sup_U |X|,$$

for some constant $C = C(W) < \infty$ and for all $X \in C^1_c(U;\mathbb{R}^{n+k})$ with $\mathrm{spt}X \subset W$. In this case, it follows from the Riesz representation theorem, see Simon [51], that the total variation measure of δV, denoted $\|\delta V\|$, is a well-defined Radon measure on U characterised by

$$\|\delta V\|(W) = \sup |\delta V(X)|,$$

where the supremum is taken over all $X \in C_c^1(U; \mathbb{R}^{n+k})$ with $|X| \leq 1$ everywhere and $\mathrm{spt} X \subset\subset W$.

In fact the Riesz representation theorem implies further that there is a $\|\delta V\|$-measurable function ν, with $|\nu| = 1$ $\|\delta V\|$-almost everywhere, such that

$$\delta V(X) = \int_{G_n(U)} \mathrm{div}_S X(x) \mathrm{d}V(x, S) = -\int_U X(x) \cdot \nu(x) \mathrm{d}\|\delta V\|(x).$$

In particular, if $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, by applying the Radon-Nikodym differentiation theorem (see Simon [51]) we may write

$$\delta V(X) = \int_{G_n(U)} \mathrm{div}_S X(x) \mathrm{d}V(x, S) = -\int_U X(x) \cdot \vec{H}(x) \mathrm{d}\|V\|(x), \qquad (2.2.2)$$

with $\vec{H} := \nu D_{\|V\|} \|\delta V\|(x)$, where $D_{\|V\|} \|\delta V\|(x)$ denotes the Radon-Nikodym derivative of $\|\delta V\|$ at x with respect to $\|V\|$. In view of the classical formula (1.1.3) we call $\vec{H}$ the *generalised mean curvature* of V.

The following property of stationary varifolds will be important later, see Simon [51] for the proof.

Theorem 2.2.14 (Constancy theorem). *Suppose that V is a stationary general n-varifold in an open set $U \subset \mathbb{R}^{n+k}$, and that* $\mathrm{spt}\|V\| \subset M$*, where M is a connected n-dimensional C^2 submanifold of U. Then V is rectifiable and $V = \mathbf{v}(M, \theta_0 \mathbb{1}_M)$ for some constant θ_0.*

2.2.3 Monotonicity formula

While general n-varifolds can be very wild, we expect stationarity to imply some level of regularity. We next introduce one of the most fundamental results concerning stationary varifolds: the monotonicity formula. It gives us control of the rate of growth of area for stationary varifolds, and also has implications for the asymptotic behaviour at singularities. Further basic regularity results will be discussed in Section 2.3.

Theorem 2.2.15 (Monotonicity formula)**.** *Suppose that V is a stationary general n-varifold in U. Then for any $x \in U$ and $0 < \sigma < \rho \leq \operatorname{dist}(x, \partial U)$ we have*

$$\frac{\|V\|(B_\rho(x))}{\rho^n} - \frac{\|V\|(B_\sigma(x))}{\sigma^n} = \int_{G_n(B_\rho(x)\setminus B_\sigma(x))} \frac{|p_{S^\perp}(y-x)|^2}{|y-x|^{n+2}} \mathrm{d}V(y,S), \qquad (2.2.3)$$

where $p_{S^\perp}$ denotes the projection to the normal space of S.

In particular, the mass ratios $\rho^{-n}\|V\|(B_\rho(x))$ are monotone non-decreasing as a function of ρ.

Remark 2.2.16. *If $V = \mathbf{v}(M,\theta)$ is rectifiable,* (2.2.3) *can be written*

$$\frac{\|V\|(B_\rho(x))}{\rho^n} - \frac{\|V\|(B_\sigma(x))}{\sigma^n} = \int_{G_n(B_\rho(x)\setminus B_\sigma(x))} \frac{|(y-x)^\perp|^2}{|y-x|^{n+2}} \mathrm{d}\|V\|(x),$$

where $(\cdot)^\perp$ denotes the orthogonal projection to $(T_xM)^\perp$ wherever T_xM exists.

Since the mass ratios are monotone, we may pass to the limit $\rho \searrow 0$, which, after a suitable renormalisation, we define to be the density at that point.

Definition 2.2.17. *Given a stationary general n-varifold V in U and $x \in U$ we define the* density *at x, denoted $\Theta(\|V\|, x)$ to be*

$$\Theta(\|V\|, x) := \lim_{\rho \searrow 0} \frac{\|V\|(B_\rho(x))}{\omega_n \rho^n}, \qquad (2.2.4)$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. The ratios on the right correspond to the mass of the varifold in the ball $B_\rho(x)$ normalised by that of a multiplicity 1 *plane through the centre of the same ball.*

One can check that if $V = \mathbf{v}(M,\theta)$ is rectifiable, then at a point x where an approximate tangent plane exists, we have

$$\lim_{\rho \searrow 0} \frac{\|V\|(B_\rho(x))}{\omega_n \rho^n} = \lim_{\rho \searrow 0} \frac{1}{\omega_n \rho^n} \int_{B_\rho(x)} \theta(y) \mathrm{d}\mathcal{H}^n(y) = \theta(x).$$

Hence, for any stationary rectifiable n-varifold $V = \mathbf{v}(M,\theta)$, it follows that $\Theta(\|V\|, \cdot) = \theta(\cdot)$ $\mathcal{H}^n$-almost everywhere. Hence we can choose $\Theta(\|V\|, \cdot)$ as a canonical representative for $\theta(\cdot)$.

Lemma 2.2.18 (Upper semi-continuity of density)**.** *Suppose that V^j for $j \geq 1$ and V are stationary general n-varifolds in an open set $U \subset \mathbb{R}^{n+k}$ with $V^j \to V$,* *and that $x_j \in U$ with $x_j \to x \in U$. Then*

$$\limsup_{j\to\infty} \Theta(\|V^j\|, x_j) \le \Theta(\|V\|, x)$$

Proof. Fix sequences $\{x_j\} \subset U$ with $x_j \to x \in U$ and V^j stationary with $V^j \to V$. Let $\varepsilon > 0$. Since $\|V^j\| \to \|V\|$ as Radon measures and the V^j and V are all stationary, it follows that for $\rho > 0$ sufficiently small and j sufficiently large we have

$$\begin{aligned}
\Theta(\|V\|, x) + \varepsilon &\ge \frac{\|V\|(B_\rho(x))}{\omega_n \rho^n} + \frac{\varepsilon}{2} \\
&\ge \frac{\|V^j\|(B_\rho(x))}{\omega_n \rho^n} \\
&\ge \frac{\|V^j\|(B_{\rho - |x_j - x|}(x_j))}{\omega_n (\rho - |x_j - x|)^n} \frac{(\rho - |x_j - x|)^n}{\rho^n} \\
&\ge \Theta(\|V^j\|, x_j) \frac{(\rho - |x_j - x|)^n}{\rho^n}.
\end{aligned}$$

Taking the lim sup of both sides and letting $\varepsilon \searrow 0$ we get the result. □

Remark 2.2.19. *One can show that the density $\Theta(\|V\|, \cdot)$ exists $\|V\|$-almost everywhere in U if the first variation of V is merely locally bounded in U. This follows from the Radon-Nikodym differentiation theorem and the addition of a suitable exponential factor in the monotonicity formula, see Simon [51] for details.*

2.2.4 Compactness theorems

General varifolds, being Radon measures on a Grassmann bundle and hence dual to compactly supported continuous functions on the Grassmann bundle, inherit compactness properties automatically from the Banach-Alaoglu Theorem (see for example [6]). Indeed given a bounded sequence of general n-varifolds V^j there exists a subsequence $V^{j'}$ and limit V such that $V^{j'} \to V$ in the weak* sense of measures. That is to say given any $f \in C_c(G_n(U))$ we have

$$\lim_{j\to\infty} \int_{G_n(U)} f(x,S) \mathrm{d}V^{j'}(x,S) = \int_{G_n(U)} f(x,S) \mathrm{d}V(x,S).$$

Notice in particular that $(x,S) \mapsto \mathrm{div}_S X(x)$ is a valid test function for any $X \in C_c^1(U; \mathbb{R}^{n+k})$. Thus a converging sequence of stationary general n-varifolds must converge to a stationary general n-varifold. General n-varifolds however have very bad a priori regularity, and so when working on problems in geometry it is desirable and far more natural to work with rectifiable n-varifolds. Hence in order to establish a good compactness theorem for rectifiable n-varifolds, we need a rectifiability lemma that will allow us to conclude, for example, that under certain reasonable assumptions the space of stationary rectifiable n-varifolds is closed under convergence of general n-varifolds. Such a rectifiability lemma was originally proved by Allard [1] (see also [51]) and is stated as follows.

Theorem 2.2.20. *Suppose that V has locally bounded first variation in U, and that $\Theta(\|V\|, x) > 0$ for $\|V\|$-almost every $x \in U$. Then V is a rectifiable n-varifold and so for some countably n-rectifiable set M and locally $\mathcal{H}^n$-integrable $\theta\colon M \to (0, \infty)$ we may write $V = \mathbf{v}(M, \theta)$.*

Combining this with the compactness of general n-varifolds inherited from their definition as Radon measures, as well as upper-semicontinuity of the density we arrive at the following compactness theorem.

Theorem 2.2.21 (Compactness)**.** *Suppose that V^j is any sequence of rectifiable n-varifolds in U satisfying*

$$\sup_{j\geq 1}(\|V^j\|(W) + \|\delta V^j\|(W)) < \infty,$$

for each $W \subset\subset U$, and that $\Theta(\|V^j\|, x) \geq 1$ on $U \setminus A_j$ for some sequence of sets $A_j \subset U$ with $\|V^j\|(A_j \cap W) \to 0$ as $j \to \infty$ for every $W \subset\subset U$.

Then there is a subsequence $V^{j'}$ and a rectifiable n-varifold V with locally bounded first variation in U such that $V^{j'} \to V$ in the sense of varifolds (i.e. weak convergence of Radon measures on $G_n(U)$), $\Theta(\|V\|, x) \geq 1$ for $\|V\|$-almost every $x \in U$ and*

$$\|\delta V\|(W) \leq \liminf_{j'\to\infty} \|\delta V^{j'}\|(W)$$

for every $W \subset\subset U$.

Remark 2.2.22. *Allard [1] showed, in addition to the above, that if each of the V^j is integral, then so is the limit V. In this case the density lower bound hypothesis is trivially satisfied along the sequence.*

Notice also that if each of the V^j is stationary, then so is the limit V, and furthermore one only needs local mass bounds along the sequence, as the first variation measure is zero. We summarise this in the following corollary.

Corollary 2.2.23. *Suppose that V^j is a sequence of stationary integral n-varifolds in an open set $U \subset \mathbb{R}^{n+k}$, and that*

$$\sup_{j \geq 1} \|V^j\|(W) < \infty,$$

for every $W \subset\subset U$. Then there is a subsequence $V^{j'}$ and a stationary integral n-varifold V such that $V^{j'} \to V$ in the sense of varifolds.

2.2.5 Tangent cones

As mentioned previously, one of the primary reasons for working in the space of varifolds is that we would like to conduct a blow-up analysis at singular points of minimal surfaces. The monotonicity formula and compactness theorem together imply that sequences of rescalings at a point will converge subsequentially to some limit varifold. The structure of these limits, i.e. the asymptotics of the varifold, can be used to prove local regularity properties at that point.

We recall the function $\eta_{x,\rho}$, defined by

$$\eta_{x,\rho}(y) := \frac{y - x}{\rho}.$$

Suppose that V is a stationary integral n-varifold. Let $\rho_j \in (0,1)$ for $j \geq 1$ satisfy $\rho_j \searrow 0$, and let $x \in \operatorname{spt}\|V\|$. Then it follows from the monotonicity formula that the sequence $V^j := \eta_{x,\rho_j \#} V$ has locally bounded mass, indeed for any $R > 0$ and j sufficiently large we have

$$\frac{\|V^j\|(B_R(0))}{\omega_n R^n} = \frac{\|V\|(B_{R\rho_j}(x))}{\omega_n (R\rho_j)^n} \leq \frac{\|V\|(B_1(x))}{\omega_n}.$$

Hence by the compactness theorem, in particular Corollary 2.2.23, it follows that $V^j \to C$ in the varifold sense for some stationary integral n-varifold C in $\mathbb{R}^{n+k}$. The monotonicity formula implies

$$\frac{\|C\|(B_\rho(0))}{\omega_n \rho^n} \equiv \Theta(\|V\|, x)$$

for every $\rho \in (0, \infty)$. Applying the monotonicity formula again we have

$$\int_{B_\rho(0)} \frac{|y^\perp|^2}{|y|^{n+2}} \mathrm{d}\|C\|(y) = 0$$

for every $\rho \in (0, \infty)$. This in particular implies that $y^\perp = 0$ for $\|C\|$-almost every $y \in \operatorname{spt}\|C\|$, from which one can deduce that C is a cone, i.e. that $\eta_{0,\rho\#}C = C$ for every $\rho \in (0, \infty)$. See Simon [51] for the details.

Definition 2.2.24. *Given a stationary integral n-varifold V in U, and a sequence $\rho_j \searrow 0$ we call any subsequential limit of the sequence $\eta_{x,\rho_j\#}V$ a* tangent cone at *x. We denote by* $\mathrm{VarTan}(V, x)$ *the set of all tangent cones of V at $x \in U$.*

Remark 2.2.25. *Notice in particular that we allow for the possibility that there could be multiple distinct tangent cones at any point. Indeed one cannot rule out a priori that different sequences of rescalings could produce different limits. The uniqueness of tangent cones has thus far only been established in various special circumstances. For a list of some of the known results, see [65].*

2.3 Regularity theory

In this section we will be interested primarily in rectifiable n-varifolds, which we refer to simply as varifolds or n-varifolds. As has been observed already, the introduction of the notion of varifolds has given us access to compactness theorems which are useful tools when doing analysis. To obtain these we have sacrificed a large amount of a priori regularity. One would expect however that stationary varifolds would exhibit better than worst case regularity in general, in part because fast oscillations or jagged corners, which contribute to the singular structure, are somehow wasteful of area. In this section we aim to put these heuristic arguments on a firmer footing, and introduce the main tools that are used in the regularity theory of stationary varifolds. To begin with we formalise the definition of the regular set and the singular set that was given in Section 1.2.

Definition 2.3.1. *Given a varifold V we define the* regular part *of V, denoted* $\mathrm{reg}V$*, to be all points $x \in \operatorname{spt}\|V\|$ for which we can find an open set U with $x \in U$ such that $\operatorname{spt}\|V\| \cap U$ is a C^1 submanifold of U containing x. We define the* singular set*, denoted* $\mathrm{sing}V$*, to be the set of all points $x \in \operatorname{spt}\|V\|$ such that $x \notin \mathrm{reg}V$.*

2.3.1 Allard's Theorem

One of the first major steps in the regularity theory of integral varifolds was the following regularity theorem of Allard [1]. Allard assumes only mass bounds, stationarity and L^2-closeness to a plane in the unit ball, and is able to conclude that the support of the varifold, in the interior, consists of a $C^{1,\alpha}$ graph with estimates. Since Allard's seminal paper, there hasn't been much progress on the general regularity of stationary n-varifolds. See Section 1.2 for a discussion of some of the known results that have been proved in the presence of additional assumptions.

Theorem 2.3.2 (Allard regularity)**.** *Suppose that $V = \mathbf{v}(M, \theta)$ is stationary in $B_1(0) \subset \mathbb{R}^{n+k}$ and that $0 \in \operatorname{spt}\|V\|$, $\Theta(\|V\|, x) \geq 1$ for every $x \in \operatorname{spt}\|V\|$. Given α, $\delta \in (0, 1)$, there exists $\varepsilon = \varepsilon(n, k, \alpha, \delta) \in (0, 1)$ such that if*

(1)

$$\frac{\|V\|(B_1(0))}{\omega_n} < 2 - \delta, \quad \textit{and}$$

(2)

$$\int_{B_1(0)} \operatorname{dist}^2(x, P) \mathrm{d}\|V\|(x) < \varepsilon$$

for some n-dimensional subspace $P \subset \mathbb{R}^{n+k}$,

then there is $\beta = \beta(n, k, \alpha, \delta) \in (0, 1)$ such that $\operatorname{spt}\|V\| \cap B_\beta(0) = \operatorname{graph}(u) \cap B_\beta(0)$ where $u \colon P \to P^\perp$ is a $C^{1,\alpha}$ function satisfying the estimate

$$\|u\|_{C^{1,\alpha}(B_\beta(0))} \leq C \left(\int_{B_1(0)} \operatorname{dist}^2(x, P) \mathrm{d}\|V\|(x) \right)^{1/2},$$

where $C = C(n, k, \alpha, \delta)$.

Remark 2.3.3. *1) By standard Schauder theory for the solutions of elliptic partial differential equations, the function u is in fact smooth, with estimates on the derivatives of any order.*

2) The theorem implies that if $V = \mathbf{v}(M, \theta)$ is stationary, $\theta \geq 1$ and also $\theta < 2$ $\mathcal{H}^n$-almost everywhere, then any point x where $T_x M$ exists is regular. Indeed at such a point we have $\Theta(\|V\|, x) = \theta(x) < 2 - \delta$ for some $\delta = \delta(x) > 0$. The monotonicity formula then implies that at small scales the mass ratios *of balls centered at x are less than $2 - \delta/2$ say, and the existence of T_xM implies that at small scales V is also L^2-close to a plane (i.e. T_xM). Thus picking a scale small enough such that both of these are true, then rescaling to unit scale and translating to the origin we can satisfy all the assumptions of Allard's theorem.*

3) *It is currently an open question as to whether $\mathcal{H}^n(\mathrm{sing}V) = 0$ with only the assumption that $\theta \geq 1$. The answer is yes in codimension 1 if we assume in addition that V is stable, due to work of Wickramasekera [64], and also in any codimension if we assume V is area-minimising, due to work of Almgren [4].*

4) *It is possible to prove versions of the monotonicity formula and also Allard's theorem if we relax the assumption of stationarity and instead assume only that the varifold has generalised mean curvature $\vec{H} \in L^p_{loc}(U)$ for some $p > n$. In this case one must take $\alpha = 1 - n/p \in (0,1)$. The fact that we make take any $\alpha \in (0,1)$ for V stationary follows from the fact that $\vec{H} \equiv 0$ of course implies $\vec{H} \in L^p_{loc}(U)$ for any $p \geq 1$.*

Corollary 2.3.4. *If V is stationary in some open set $U \subset \mathbb{R}^{n+k}$, and we have $\Theta(\|V\|, x) \geq 1$ for all $x \in \mathrm{spt}\|V\| \cap U$, then $\mathrm{reg}V$ is open and dense in $\mathrm{spt}\|V\| \cap U$. In particular, $\mathrm{sing}V$ is closed and nowhere dense.*

Proof. Suppose that $y \in \mathrm{spt}\|V\| \cap U$ and $0 < \rho < \mathrm{dist}(y, \partial U)$. Define

$$\alpha := \inf_{x \in \mathrm{spt}\|V\| \cap B_\rho(y)} \Theta(\|V\|, x),$$

and note that $\alpha \geq 1$. Choose $z \in \mathrm{spt}\|V\| \cap B_\rho(y)$ such that $\Theta(\|V\|, z) < 3\alpha/2$, which we may do by definition of α. Then by upper semi-continuity of the density, it follows that $\alpha \leq \Theta(\|V\|, x) < 3\alpha/2$ for every $x \in \mathrm{spt}\|V\| \cap B_\sigma(z)$ for some suitable $\sigma > 0$. Therefore we may define

$$V_1 := \mathbf{v}(\mathrm{spt}\|V\| \cap B_\sigma(z), \alpha^{-1}\Theta(\|V\|, \cdot)|_{\mathrm{spt}\|V\| \cap B_\sigma(z)}),$$

and we find that V_1 is stationary in $B_\sigma(z)$ and satisfies $1 \leq \theta(x) < 2$ for every $x \in \mathrm{spt}\|V_1\|$. Consequently by part 2) of the preceding remark we find $\mathcal{H}^n(\mathrm{sing}V_1) = 0$, from which the conclusions readily follow. □

2.3.2 Stratification of the singular set

It is possible to stratify the singular set of a stationary varifold in the following simple but powerful way.

Definition 2.3.5. *Given a stationary cone C, we define the* spine *of C, $S(C)$, to be the set of points*

$$S(C) := \{z \in \mathbb{R}^{n+k} \mid \Theta(\|C\|, z) = \Theta(\|C\|, 0)\}.$$

It is simple to show using the monotonicity formula (see for example Simon [53]) and upper semi-continuity of density, that $\Theta(\|C\|, z) \leq \Theta(\|C\|, 0)$ for any $z \in \mathbb{R}^{n+k}$. Moreover one can show that if $\Theta(\|C\|, 0) = \Theta(\|C\|, z)$ then $\mathrm{spt}\|C\|$ is translation invariant in the z direction, and hence that $S(C)$ is a subspace of $\mathbb{R}^{n+k}$. Given a stationary varifold V we define

$$\mathcal{S}_j := \{x \in \mathrm{sing}V \mid \dim S(C) \leq j \;\text{ for all }\; C \in \mathrm{VarTan}(V, x)\},$$

that is, $\mathcal{S}_j$ consists of all points in the singular set at which no tangent cone C can be written $C = R_{\#}(C_0 \times \mathbb{R}^{j+1})$, where R is a rotation. The following lemma was first established by Almgren [4], and is itself a refinement of the dimension reduction principle of Federer [18], see also Simon [51]. Analogous results for energy minimising maps and mean curvature flow have been proved by Simon [53] and White [61] respectively.

Lemma 2.3.6. *For each $j = 0, 1, \ldots, n$, we have*

$$\dim_{\mathcal{H}} \mathcal{S}_j \leq j.$$

In particular this lemma tells us that if one wishes to prove some property holds except on a set of small dimension, one needs only to classify the simplest cones, i.e. those with low-dimensional cross section, and analyse behaviour at points where those tangent cones occur. It is not necessary to completely classify all cones.

In our case, we will use Lemma 2.3.6 slightly differently to show an abundance of points with a density lower bound. Indeed, suppose V is a stationary varifold and that $S \subset \mathrm{sing}V \subset \mathrm{spt}\|V\|$ has positive $\mathcal{H}^{n-1}$-measure. Then Lemma 2.3.6

implies that $\mathcal{H}^{n-1}$-almost every point of S has a tangent cone that is either a plane with multiplicity at least 2, or has a one dimensional cross section. In the latter case, the cross section must consist of at least three half lines meeting at a point. If one can rule out the possibility that the cross section is exactly three half-lines, then once again the multiplicity will be at least 2, and so $\mathcal{H}^{n-1}$-almost every point of S has density at least 2. The existence of many 'good density points' will later be important as they are precisely where the main L^2 estimates hold.

2.4 Two-valued functions

A crucial step in establishing the required L^2 estimates is establishing the graphical approximation lemma (Lemma 3.2.6) in Section 3.2. Allard's theorem allows us to approximate much of the supports of the varifolds we will consider as single-valued graphs over suitable planes, but this alone does not give us enough control. Instead it is necessary to also approximate some of the support as the graph of a two-valued Lipschitz function. The notion of a multi-valued function was first introduced by Almgren [4], see also the more recent work of De Lellis-Spadaro for an alternative approach [13].

Definition 2.4.1. *We denote by $\mathcal{A}_2(\mathbb{R}^n)$ the set of all unordered pairs of points in $\mathbb{R}^n$. A* two-valued function *on an open set $\Omega \subset \mathbb{R}^m$ is a function $f\colon \Omega \to \mathcal{A}_2(\mathbb{R}^n)$. We equip $\mathcal{A}_2(\mathbb{R}^n)$ with the metric $\mathcal{G}$ defined by*

$$\mathcal{G}(a,b) := \min\left\{\sqrt{|a_1-b_1|^2+|a_2-b_2|^2}, \sqrt{|a_1-b_2|^2+|a_2-b_1|^2}\right\},$$

where $a = \{a_1, a_2\}$ and $b = \{b_1, b_2\}$. We also define

$$|a| := \mathcal{G}(a, \{0,0\}) = \sqrt{|a_1|^2+|a_2|^2}. \tag{2.4.1}$$

We say that f is Lipschitz *on Ω with constant $L \geq 0$ if*

$$\mathcal{G}(f(x), f(y)) \leq L|x-y| \quad \text{for all } x,\, y \in \Omega$$

and we define

$$\mathrm{Lip}(f) := \sup\left\{ \frac{\mathcal{G}(f(x), f(y))}{|x-y|} \;\middle|\; x,\, y \in \Omega,\;\; x \neq y \right\}.$$

Furthermore, we say that f is differentiable *at $x \in \Omega$ if there exists a two-valued affine function $l_x\colon \mathbb{R}^m \to \mathcal{A}_2(\mathbb{R}^{n+k})$ of the form $l_x(y) = \{A_1^x y + b_1^x, A_2^x y + b_2^x\}$ for constant matrices A_1^x, $A_2^x \in \mathbb{R}^{m\times(n+k)}$ and constant vectors b_1^x, $b_2^x \in \mathbb{R}^m$, such that*

$$\lim_{y\to x} \frac{\mathcal{G}(f(y), l_x(y))}{|x-y|} = 0.$$

It is easy to see that l_x must be unique if it exists, in which case we define the derivative *of f at x to be the unordered pair $Df(x) := \{A_1^x, A_2^x\}$.*

It turns out that Rademacher's Theorem generalises to two-valued functions, see [4] or [13] for the proof.

Theorem 2.4.2 (Rademacher's theorem)**.** *Suppose that $\Omega \subset \mathbb{R}^m$ is open and $f\colon \Omega \to \mathcal{A}_2(\mathbb{R}^n)$ is Lipschitz, then f is differentiable almost everywhere, and $|Df(x)| \leq \mathrm{Lip}(f)$ wherever $Df(x)$ exists.*

The final result we need is the following Lipschitz approximation theorem of Almgren [4].

Theorem 2.4.3. *Suppose that α, β, $\gamma \in (0,1)$. There exists $\varepsilon \in (0,1)$ depending on n, k, α, β and γ such that if V is a stationary integral n-varifold in $B_{1+\gamma}(0)$ satisfying:*

(a) V satisfies the mass bounds

$$\frac{\|V\|(B_{1+\gamma}(0))}{\omega_n(1+\gamma)^n} \leq 3-\alpha \qquad 1+\alpha \leq \frac{\|V\|(B^n_{1+\gamma/3}(0)\times\mathbb{R}^k)}{\omega_n(1+\gamma/3)^n} \leq 3-\alpha,$$

(b) V satisfies the following height excess bound

$$\int_{B_{1+\gamma}(0)} \mathrm{dist}^2(x, \mathbb{R}^n \times \{0\}^k)\mathrm{d}\|V\|(x) \leq \varepsilon;$$

then there exists a Lipschitz two-valued function $f\colon B_1^n(0) \to \mathcal{A}_2(\mathbb{R}^k)$ and an $\mathcal{H}^n$-measurable set $\Sigma \subset B_1^n(0)$ such that:

(1) $\mathrm{Lip}(f) \leq \beta$,

(2) we have the following bounds on the measure of Σ

$$\mathcal{H}^n(\Sigma) + \|V\|(\Sigma \times \mathbb{R}^k) \leq C \int_{B_{1+\gamma}(0)} \mathrm{dist}^2(x, \mathbb{R}^n) \mathrm{d}\|V\|(x),$$

where $C = C(n, k, \alpha, \beta, \gamma)$,

(3) the support of V *coincides with the graph of* f *away from* Σ, *i.e.*

$$\mathrm{spt}\|V\| \cap \Big((B_1^n(0) \setminus \Sigma) \times \mathbb{R}^k \Big) = \mathrm{graph}|_{B_1^n(0) \setminus \Sigma}(f).$$

Chapter 3

Boundary regularity for stationary varifolds

3.1 Motivation

As discussed in Chapter 1, relatively little is known about the boundary regularity of stationary varifolds. Indeed the only known result assuming only stationarity is due to Allard [2] with refinements by Bourni [7]. Having developed the relevant terminology in Chapter 2, we now elaborate on what they were able to prove.

One of the barriers to investigating boundary regularity of stationary varifolds is that they lack a natural notion of boundary in the first place. Allard's result avoids this issue, as the only statement about the varifold's boundary values appears in the conclusion, where the support is given by a classical smooth manifold. In particular, he proved the following.

Theorem 3.1.1 (Allard '75, Bourni '14). *For each $\varepsilon \in (0,1)$, $\alpha \in (0,1)$ there is $\delta = \delta(\alpha, \varepsilon) > 0$ such that if B is the graph over $\{0\}^{k+1} \times B_4^{n-1}(0)$ of some $C^{1,\alpha}$ function w, with $\|w\|_{1,\alpha} \leq \delta$ and $w(0) = 0$, and V is a rectifiable n-varifold in $B_1(0)$, which is stationary in $B_1(0) \setminus B$, has $0 \in \operatorname{spt}\|V\|$, and satisfies $\|V\|(B_1(0)) \leq (1+\delta)/2$, then we have the following conclusions:*

1) *There is $u \in \mathbb{R}^{k+1} \times \{0\}^{n-1}$ such that V has a unique tangent half-plane at the origin given by*

$$H = \{y + tu \mid y \in \{0\}^{k+1} \times \mathbb{R}^{n-1},\ t \geq 0\}.$$

2) If P denotes the plane

$$P = \{y + tu \mid y \in \{0\}^{k+1} \times \mathbb{R}^{n-1},\ t \in \mathbb{R}\},$$

then the height excess over P is small, i.e.

$$\int_{B_1(0)} \mathrm{dist}^2(x, P) \mathrm{d}\|V\| \leq \varepsilon^2.$$

3) $M := \mathrm{spt}\|V\| \cap B_{1-\varepsilon}(0) \setminus B$ is a continuously differentiable submanifold, closed relative to $B_{1-\varepsilon}(0) \setminus B$, whose closure in $B_{1-\varepsilon}(0)$ contains $B \cap B_{1-\varepsilon}(0)$. Further more M projects injectively onto P (under the orthogonal projection to P).

4) If p_{T_xM} and p_P denote the orthogonal projections to the tangent space T_xM and the plane P respectively, then

$$|p_{T_xM} - p_P| \leq C \sup\{\varepsilon, \delta\},$$

for every $x \in \mathrm{spt}\|V\| \cap (B_{1-\varepsilon}(0) \setminus B)$.

5) If p_{T_xM} and p_{T_yM} denote the orthogonal projections to the tangent spaces T_xM and T_yM respectively, then

$$|p_{T_xM} - p_{T_yM}| \leq C \sup\{\varepsilon, \delta\} |x - y|^\alpha,$$

for every $x,\ y \in \mathrm{spt}\|V\| \cap (B_{1-\varepsilon}(0) \setminus B)$.

Note that 3) implies that in 4) and 5) we can take the classical tangent planes which exist at every point.

Remark 3.1.2. *Allard's original result required a $C^{1,1}$ boundary, as this ensured the nearest point projection to the boundary had enough regularity to be used construct test vector fields to plug into the first variation formula. Choosing vector fields carefully one can show that analogues of the first variation and the monotonicity formula hold at boundary points. The contribution of Bourni was to relax this assumption to $C^{1,\alpha}$ by constructing a new 'distance' function via a Whitney partition, which was smooth but satisfied appropriate inequalities. Using this she* *was able to rederive the boundary monotonicity formula for $C^{1,\alpha}$ boundaries, and hence generalise the proof of the regularity theorem.*

We also note that, as in the interior theorem, one can also relax the stationarity assumption to the generalized mean curvature being in L^p_{loc} for $p > n$.

The problem considered in this chapter is motivated by the following question, which arises naturally in light of the above theorem: "What can we say about the local regularity of a stationary varifold at a boundary point where there is a tangent cone consisting of two half-planes meeting along their common boundary?". We can reformulate this as the following. Suppose that B is a $C^{1,\alpha}$ curve through the origin, and that V is a stationary integral n-varifold in $B_1(0) \setminus B$. Moreover suppose that V is close in mass and in L^2 to a pair of half-planes $\mathbf{C}^{(0)}$ meeting along T_0B. Can we conclude that $\text{spt}\|V\| \cap B_\gamma(0)$ consists of two smooth sheets meeting along B for some $\gamma > 0$?

It is not hard to see that without additional assumptions, this cannot be the case. Consider the following example: let $n = 2$ and take three half-planes meeting at angles of $2\pi/3$ radians along a common line. Now take one of the half-planes and consider a $C^{1,\alpha}$ curve B which lies completely within it, passes through the origin, and is tangent to the axis along which the half-planes meet at the origin. It is possible to construct B satisfying these conditions, but such that it oscillates wildly, touching the axis at very many points. Then near the origin one can find many points where B touches the axis, and many points where B is away from the axis. Now delete everything on one side of B, so that what remains is two half-planes meeting at the origin, and a jagged piece of the third half-plane that comes out from the axis to meet B, see Figure 3.1. This will be stationary in $B_1(0) \setminus B$, and we can also ensure L^2-closeness to a pair of half-planes, but near

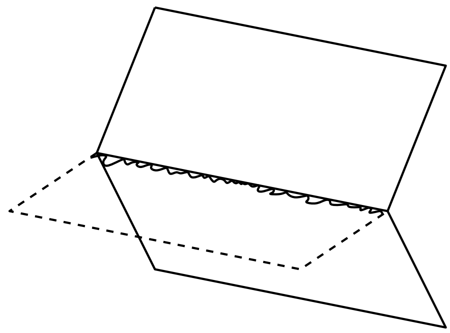

Figure 3.1: A counterexample to the desired regularity.

the origin the tangent cones at points on B can switch arbitrarily often between a single half-plane, or a pair of half-planes meeting along an $(n-1)$-dimensional subspace, so the desired regularity statement does not hold.

In the proof of his boundary regularity theorem, Allard makes crucial use of a reflection principle which allows him to apply his interior theorem. In order to mimic his arguments, one would need to develop a suitable, analogous interior theorem. In this case, that corresponds to an interior regularity theorem for a stationary varifold that is L^2-close, and close in mass, to a pair of planes intersecting along an $(n-1)$-dimensional subspace. In particular, one would like to conclude that in the interior, the support consists of four smooth sheets meeting along the subspace or a smooth curve close to the subspace. Again however there are simple counter examples. For example, take a pair of planes intersecting along an $(n-1)$-dimensional axis, and desingularise the intersection by introducing smooth 'necks' to obtain a Scherk style surface, see Figure 3.2. Such a desingularisation is guaranteed to exist by work of Kapouleas [36]. By scaling, this can be made arbitrarily close to a pair of planes, but the curvature in the neck regions blows up, so there is no hope of proving any sort of quantitative regularity properties.

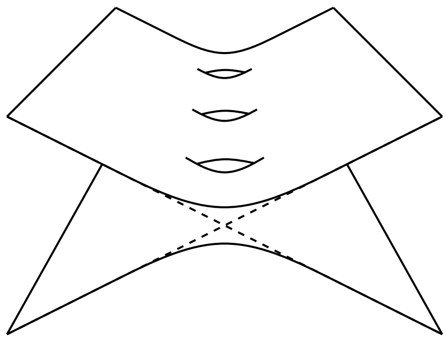

Figure 3.2: A smooth minimal surface close to a pair of intersecting planes.

In the remainder of this chapter we show that under additional assumptions which rule out the above situations, we can prove the aforementioned interior theorem. Specifically we assume the absence of triple junction singularities, i.e. points at which locally the varifold consists of three smooth sheets meeting along a common boundary, as well as the presence of 'plenty' of singularities near the axis. Thanks to an argument using the reflection principle this immediately implies a corresponding boundary regularity result. We will state the results precisely in Section 3.6, but roughly speaking our main results are stated as follows.

Theorem (Regularity theorem)**.** *Suppose that V is a stationary n-varifold in $B_1(0)$. If V is sufficiently close in L^2 and in mass to a pair of planes intersecting along an $(n-1)$-dimensional axis, and if* $\text{sing}V$ *satisfies certain structural assumptions; then in a neighbourhood of the origin* $\text{spt}\|V\|$ *consists of four smooth n-dimensional submanifolds meeting only along a common $(n-1)$-dimensional $C^{1,\alpha}$ submanifold.*

Corollary (Boundary regularity corollary)**.** *Let $B := \{0\}^{k+1} \times \mathbb{R}^{n-1}$. If V is a stationary n-varifold in $B_1(0) \setminus B$ and is sufficiently close in L^2 and in mass to a pair of half-planes meeting along B, and if* $\text{sing}V$ *satisfies the same structural assumptions as in the above theorem; then in a neighbourhood of the origin* $\text{spt}\|V\|$ *consists of two smooth sheets meeting along B, their common boundary.*

Remark 3.1.3. *It currently remains open whether the regularity theorem can be used to generalise the corollary to the case where B is instead a $C^{1,\alpha}$ submanifold with suitably small $C^{1,\alpha}$ norm.*

We prove the interior regularity result by applying the so-called blow-up method of Simon [52] with refinements due to Wickramasekera [64]. Simon studied multiplicity one classes of minimal submanifolds, and was able to prove estimates to control the linearisation of the minimal surface operator (which we refer to as the blow-up), at singularities that possess a cylindrical tangent cone. These are cones which, after rotating, can be written in the form $\mathbf{C}_0 \times \mathbb{R}^l$, where $\mathbf{C}_0$ is a stationary m-dimensional cone with an isolated singularity, and $m + l = n$. Wickramasekera has adapted these techniques to study the singularities of stable hypersurfaces, most notably in [64], and also in [62, 63].

The key ingredient in the proof of the main theorems is an 'excess improvement lemma' (see Lemma 3.5.1). This lemma says that given a stationary varifold V which is sufficiently close to a cone $\mathbf{C}^{(0)}$ consisting of a pair of planes intersecting along an $(n-1)$-dimensional subspace in L^2-distance, we can find a new cone $\mathbf{C}$, consisting of four half-planes meeting along an $(n-1)$-dimensional subspace, such that the L^2-distance to $\mathbf{C}$ at a smaller scale θ has decayed by a fixed power of θ. Iterating this lemma carefully will establish the main theorems.

3.1.1 Notation and organisation

We now introduce some of the basic notation used throughout this chapter. We work in $\mathbb{R}^{n+k}$. We will denote points $X \in \mathbb{R}^{n+k}$ with capital latin letters, and make the identification $X = (x, y)$ where $x \in \mathbb{R}^{k+1}$ and $y \in \mathbb{R}^{n-1}$. We define $B := \{0\}^{k+1} \times \mathbb{R}^{n-1}$, and we define the function $r = r(X) := |x| = |(x, 0)|$, i.e. the distance to B.

We denote by $\mathcal{C}$ the collection of all cones $\mathbf{C}$ such that $\mathrm{spt}\|\mathbf{C}\|$ consists of four half-planes meeting along B, and $\Theta(\|\mathbf{C}\|, X) = 1$ for all $X \in \mathrm{spt}\|\mathbf{C}\| \setminus B$. We let H_i for $i = 1, \ldots, 4$ denote the half-planes making up $\mathbf{C}$. For any $\mathbf{C} \in \mathcal{C}$ we necessarily have the decomposition $\mathbf{C} = \mathbf{C}_0 \times \mathbb{R}^{n-1}$, where $\mathrm{spt}\|\mathbf{C}_0\|$ consists of four half-lines meeting at 0.

We denote by $R_0 = R_0(n) > 0$ a fixed radius that is to be chosen later. Given an n-varifold V in $B_{R_0}(0)$ we define the following two kinds of height excess

$$Q_V(\mathbf{C}) := \left(\int_{B_1 \setminus (B^{k+1} \times \mathbb{R}^{n-1})} \mathrm{dist}^2(X, \mathrm{spt}\|V\|) \mathrm{d}\|\mathbf{C}\| \right. \\ \left. + \int_{B_{R_0}} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|) \mathrm{d}\|V\| \right)^{1/2},$$

and

$$E_V(\mathbf{C}) := \left(\int_{B_1} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|) \mathrm{d}\|V\| \right)^{1/2}.$$

We also define the following open neighbourhoods of subsets of a half-plane, which we refer to as β-conical neighbourhoods.

Definition 3.1.4. *Let H be an n-dimensional half-plane with boundary B and denote by p the orthogonal projection onto the n-dimensional plane containing H. We define the β-*conical neighbourhood* of a relatively open subset $U \subset H$ to be the set*

$$C_H(U, \beta) := \{(x, y) \in \mathbb{R}^{n+k} \mid |p_\perp((x, 0))| < \beta |p((x, 0))|,\ p(x, y) \in U\}.$$

In particular $C(H, \beta) := C_H(H, \beta)$ is an open 'wedge' containing H, consisting of all points x whose distance from H is at most β times the distance of $p(x)$ to the axis B. Given a relatively open subset $U \subset H$, $C_H(U, \beta)$ is simply $C(H, \beta)$

intersected with $\bigcup_{x \in U}(x + H^{\perp})$.

Given a varifold V, and cones $\mathbf{C}$, $\mathbf{C}^{(0)} \in \mathcal{C}$, we will often assume the following for appropriately chosen $\varepsilon_A \in (0,1)$ and $\delta_A \in (0, 1/4]$.

Hypotheses A. *1)* $\mathbf{C}, \mathbf{C}^{(0)} \in \mathcal{C}$ *with* $\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\mathbf{C}^{(0)}\| \cap B_1, \mathrm{spt}\|\mathbf{C}\| \cap B_1) \leq \varepsilon_A$.

2) V *is a stationary* n*-varifold in* $B_{R_0}(0)$ *with*

$$\frac{\|V\|(B_{R_0}(0))}{\omega_n {R_0}^n} \leq 2 + \delta_A.$$

3) V *satisfies* $Q_V(\mathbf{C}^{(0)}) \leq \varepsilon_A$.

Remark 3.1.5. *Note that we could also have used* L^2 *distance in part 1) of Hypotheses A, since the definition of* $\mathcal{C}$ *implies that these two notions of distance are Lipschitz equivalent.*

In later chapters we will need to make certain structural assumptions on the singular set.

Definition 3.1.6. *We denote by* $\mathcal{V}$ *the class of all* n*-varifolds* V *in* $B_{R_0}(0)$ *satisfying the following.*

(M1) V *has no triple junction singularities in* $B_1(0) \setminus B$.

(M2) The orthogonal projection of $\mathrm{sing} V \cap B_1$ *to* B *has full* $\mathcal{H}^{n-1}$*-measure.*

Remark 3.1.7. *Any varifold arising as the limit of smooth submanifolds cannot have any triple junction singularities, so for such a* V *condition (M1) is automatically satisfied.*

The structure of the remainder of this chapter is the following. In Section 3.2 we establish graphical approximation results which imply that away from the axis, the support of V must be a smooth graph; in Section 3.3 we prove analogues of Simon's L^2 estimates; Section 3.4 contains the construction of blow-ups and the proofs of their regularity properties; Section 3.5 contains the proof of the crucial excess decay lemma; and finally Section 3.6 contains the proofs of the main regularity theorem and the boundary regularity corollary; finally in Section 3.7 we construct a cover that is of crucial importance in Section 3.2.

3.2 Graphical approximation

In this section we prove results that allow us to parametrise much of $\operatorname{spt}\|V\|$ as the graph of either a smooth single-valued function or a Lipschitz two-valued function. Rescalings of the single-valued function will be used later to construct blow-ups, whereas the two-valued functions play an important role in proving the main L^2 estimates of the next section. Crucial in all of this is the construction of a particular covering by toroidal regions, with bounded intersections. Following Simon [52], we first introduce the following notation.

Definition 3.2.1. *Given ρ, $\kappa > 0$ and $\zeta \in \mathbb{R}^{n-1}$, and with $\gamma < 1$ fixed, we define*

$$T_{\rho,\kappa}(\zeta) := \left\{ (x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-1} \,\middle|\, (|x| - \rho)^2 + |y - \zeta|^2 < \frac{\kappa^2(1-\gamma)^2\rho^2}{4} \right\}.$$

Let $\mathbf{C} \in \mathcal{C}$. We let H_i for $i = 1, \ldots, 4$ denote the four open half-planes making up $\operatorname{spt}\|\mathbf{C}\| \setminus B$, and define $D^i_{\rho,\kappa}(\zeta) := T_{\rho,\kappa}(\zeta) \cap H_i$ for $i = 1, \ldots, 4$. We also define $U^i_{\rho,\kappa}(\zeta)$ to be the open ball centred on H_i whose intersection with H_i is precisely $D^i_{\rho,\kappa}(\zeta)$.

Remark 3.2.2. *Note that $D^i_{\rho,\kappa}(\zeta)$ is a disk in H_i with radius $\kappa(1-\gamma)\rho/2$.*

We then are able to construct a cover with the following properties.

Lemma 3.2.3. *Given $c \leq 1$, $\gamma < 1$ it is possible to choose $(\xi_i, \zeta_i) \in B_1(0) \setminus B$ for $i \in \mathbb{N}$ such that $T_{|\xi_i|,c}(\zeta_i) \subset B_1(0) \setminus B$ for each i, $(T_{|\xi_i|,2c/9}(\zeta_i))$ are disjoint, and $(T_{|\xi_i|,c/2}(\zeta_i))$ cover $B_\gamma(0) \setminus B$. Moreover there is $N = N(n)$ such that $(T_{|\xi_i|,c/2}(\zeta_i))$ can be divided into $N(n)$ disjoint subcollections.*

The proof is straightforward. One simply chooses a maximal disjoint collection of tori of the form $T_{|\xi_i|,2c/9}(\zeta_i)$, then checks that the conditions all hold for this collection. We provide the details in Section 3.7. The cover $(T_{|\xi_i|,c/2}(\zeta_i))$ constructed above will be employed several times in that which follows. Before we state the graphical approximation lemmas, we choose a good value for c. Note that there is a $\beta_{\mathbf{C}^{(0)}} = \beta_{\mathbf{C}^{(0)}}(\mathbf{C}^{(0)}) \leq 1$ such that $C(H_i^{(0)}, 3\beta_{\mathbf{C}^{(0)}})$ are pairwise disjoint. Assuming $\varepsilon_{\mathbf{C}^{(0)}} = \varepsilon_{\mathbf{C}^{(0)}}(\mathbf{C}^{(0)})$ is small enough and $\mathbf{C} \in \mathcal{C}$ is such that $\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}\| \cap B_1,\ \operatorname{spt}\|\mathbf{C}^{(0)}\| \cap B_1) \leq \varepsilon_{\mathbf{C}^{(0)}}$, we have $C(H_i, 2\beta_{\mathbf{C}^{(0)}})$ are pairwise disjoint also. Thus we may choose $c = c(\mathbf{C}^{(0)}) \leq 1$ small such that $U^i_{|\xi|,c}(\zeta) \subset C(H_i, 2\beta_{\mathbf{C}^{(0)}})$. Hence we have $U^i_{|\xi_1|,c}(\zeta_1) \cap U^j_{|\xi_2|,c}(\zeta_2) = \emptyset$ for any $(\xi_1, \zeta_1) \in H_i$, $(\xi_2, \zeta_2) \in H_j$ with $i \neq j$.

In Hypotheses A we assume that the mass ratios of V in the ball B_{R_0} are bounded by $2+\delta$. Later we want to apply estimates on balls that are not centred at the origin. To do so we need the mass ratios of these balls to be bounded also. The first lemma shows us that provided we choose R_0 big enough initially, in a way that depends only on n and δ, we can ensure that all balls with centres close to the origin enjoy good mass ratio bounds also.

Lemma 3.2.4. *Let $\delta_0 > 0$. There exists $R_0 = R_0(n, \delta_0) \geq 2$ such that if $\delta \geq \delta_0$, V is stationary in $B_{R_0}(0)$ and satisfies*

$$\frac{\|V\|(B_{R_0}(0))}{\omega_n R_0^n} \leq 2 + \delta,$$

then for any $x \in B_1(0)$ and any $\rho \in (0, R_0 - |x|)$ we have

$$\frac{\|V\|(B_\rho(x))}{\omega_n \rho^n} \leq 2 + 2\delta.$$

Proof. Using the monotonicity formula we have

$$\begin{aligned} \frac{\|V\|(B_\rho(x))}{\omega_n \rho^n} &\leq \frac{\|V\|(B_{R_0-|x|}(x))}{\omega_n (R_0 - |x|)^n} \\ &\leq \frac{\|V\|(B_{R_0}(0))}{\omega_n R_0^n} \frac{R_0^n}{(R_0 - |x|)^n} \\ &\leq (2+\delta) \frac{1}{(1 - R_0^{-1})^n}. \end{aligned}$$

Evidently, the result will follow if R_0 is chosen suitably large depending on n that $(1 - R_0^{-1})^{-n} \leq (2+2\delta)/(2+\delta)$. Since $x \mapsto (2+2x)/(2+x)$ is monotone increasing for $x > 0$, it follows that provided

$$\frac{1}{(1 - R_0^{-1})^n} \leq \frac{2 + 2\delta_0}{2 + \delta_0},$$

we have the desired result. □

Henceforth, R_0 will always denote $R_0(n, 1/64)$ from the above Lemma. We next show that given V, $\mathbf{C}$ and $\mathbf{C}^{(0)}$ satisfying Hypotheses A with appropriately chosen constants, we can parametrise $\operatorname{spt}\|V\|$ as a smooth single-valued graph over $\mathbf{C}$ away from B.

Lemma 3.2.5. *Let $\mathbf{C}^{(0)} \in \mathcal{C}$ and let $\beta, \tau \in (0,1)$. There exists $\varepsilon_0 = \varepsilon_0(\mathbf{C}^{(0)}, \beta, \tau)$ such that if V, $\mathbf{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_0$ and $\delta_A = 1/4$, then there is $u \in C^2(B_{7/4}(0) \cap \operatorname{spt}\|\mathbf{C}\| \setminus (B^{k+1}_{\tau/4}(0) \times \mathbb{R}^{n-1}); (\operatorname{spt}\|\mathbf{C}\|)^{\perp})$ such that*

$$\operatorname{spt}\|V\| \cap B_{3/2}(0) \setminus (B^{k+1}_{\tau/2}(0) \times \mathbb{R}^{n-1}) \subset \operatorname{graph}(u) \subset \operatorname{spt}\|V\|,$$

and u satisfies the estimate

$$\sup r^{-1}|u| + \sup |\nabla u| \leq \beta, \tag{3.2.1}$$

where $r(X) := \operatorname{dist}(X, B)$ as defined at the start of the chapter.

Proof. Fix $\mathbf{C}^{(0)} \in \mathcal{C}$ and choose sequences V^i, $\mathbf{C}^i$, $\varepsilon_i \searrow 0$ with V^i, $\mathbf{C}^i$ and $\mathbf{C}^{(0)}$ satisfying Hypotheses A with $\varepsilon_A = \varepsilon_i$ and $\delta_A = 1/4$. We show that the conclusion holds along a subsequence, which will establish the claim. Since V^i are all stationary and have uniformly bounded mass, after passing to a subsequence we have $V^i \to V$ as varifolds, where V is integral and stationary in B_{R_0} (see Corollary 2.2.23). Moreover $Q_V(\mathbf{C}^{(0)}) = 0$, so $\operatorname{spt}\|V\| \subset \operatorname{spt}\|\mathbf{C}^{(0)}\|$ and $\operatorname{spt}\|V\| \cap H_i^{(0)} \neq \emptyset$ for each $i = 1, \ldots, 4$. The constancy theorem (Theorem 2.2.14) then implies that $\operatorname{spt}\|V\| = \operatorname{spt}\|\mathbf{C}^{(0)}\|$ with constant multiplicities θ_i on each H_i. Moreover, convergence of mass implies the mass bound passes to the limit and so

$$\frac{\|V\|(B_{R_0}(0))}{\omega_n R_0^n} \leq 2 + \frac{1}{4}.$$

Since $\operatorname{spt}\|V\| = \operatorname{spt}\|\mathbf{C}^{(0)}\|$ consists of 4 half-planes, the mass ratios must be a multiple of $1/2$. Hence we deduce $\theta_i \equiv 1$ for each i. Therefore by Allard's Regularity Theorem (see Theorem 2.3.2, also [1] or [51]) we see that we must get smooth convergence of the V^i to $\mathbf{C}^{(0)}$ in $\{|x| > \tau/8\} \cap B_{15/8}$. Moreover, the $\mathbf{C}^i$ clearly converge smoothly to $\mathbf{C}^{(0)}$ also, and so for all sufficiently large i, the set $\operatorname{spt}\|V\| \cap \{|x| > \tau/2\} \cap B_{3/2}$ is contained in the graph of a function $u \in C^2(\{|x| > \tau/4\} \cap B_{7/4} \cap \operatorname{spt}\|\mathbf{C}\|; (\operatorname{spt}\|\mathbf{C}\|)^{\perp})$, with u satisfying the estimate (3.2.1). □

The next lemma is crucial. It builds on Lemma 3.2.5 by giving a much more precise description of the behaviour of V near B. Simon in [52] proves an analogous result (cf. [52, Lemma 2.6]) by dividing $B_\gamma(0)$, with $\gamma \in (0,1)$, into toroidal regions. He then argues that in any given torus either the height excess is small and one has a graphical representation in the interior, or the excess is large, in which case an argument using the monotonicity formula can be used to bound the L^2-norm of the distance to the axis by the height excess. Here he makes crucial use of the fact that he is working in a compact multiplicity one class, which means that small height excess implies local graphicality.

In our setting, we do not rule out higher multiplicity regions a priori, and indeed the mass bounds are not restrictive enough to rule out multiplicity two regions near B. Consequently we do not have the same dichotomy employed by Simon. Indeed it is possible that there are toroidal regions with small height excess, but in which there is not a smooth graphical representation in the interior. In such regions, we apply Almgren's Lipschitz approximation theorem (Theorem 2.4.3), to parametrise large parts of V by a Lipschitz, two-valued graph. Here we mean large in the sense that one has estimates on the measure of the symmetric difference of the graph and V in terms of the height excess. We repeat this argument in every toroidal region with small height excess and large mass. Since the sets on which the graphs coincide with the varifold are not necessarily open, it is not possible to use a unique continuation argument to piece them together into a single two-valued function. Instead we need to sum over all of the graph functions, which we may do while only picking up a constant factor thanks to the cover constructed in Lemma 3.2.3.

This Lemma represents the bulk of the original contribution to the methods of Simon and Wickramasekera present in this chapter.

Lemma 3.2.6. *Let $\mathbf{C}^{(0)} \in \mathcal{C}$ and γ, β, $\tau \in (0,1)$ with $\tau \leq (1-\gamma)/10$ and $\beta \leq \beta_0$ where β_0 is as defined above. Then there exists $\varepsilon_0 \in (0,1]$ depending on $\mathbf{C}^{(0)}$, γ, β and τ such that the following holds. If V, $\mathbf{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_0$ and $\delta_A = 1/4$ then there exist relatively open sets $U = U^{(1)} \cup U^{(2)} \subset \operatorname{spt}\|\mathbf{C}\| \cap B_2(0)$ such that*

$$(x,y) \in U \;\Rightarrow\; (\tilde{x},y) \in U \;\textit{ for all }\; (\tilde{x},y) \in \operatorname{spt}\|\mathbf{C}\| \;\textit{ with }\; |\tilde{x}| = |x|,$$
$$\Big\{(x,y) \in \operatorname{spt}\|\mathbf{C}\| \cap B_{3/2} \;\Big|\; |x| > \tau\Big\} \subset U^{(1)}, \tag{3.2.2}$$

and $U^{(2)}$ is the countable union of disks D_i which can be subdivided into at most $N(n)$ pairwise disjoint subcollections. Further there exists a twice continuously differentiable function $u \in C^2(U^{(1)}; (\mathrm{spt}\|\mathbf{C}\|)^\perp)$ with

$$\mathrm{spt}\|V\| \cap B_\gamma(0) \cap \Big\{(x,y) \,\Big|\, |x| > \tau\Big\} \subset \mathrm{graph}(u) \subset \mathrm{spt}\|V\|,$$
$$\sup \frac{|u|}{r} + \sup |\nabla u| \leq \beta,$$

where $r(x,y) = |x|$ as before. Moreover if $U^{(2)} = \bigcup_i D_i$, then for each i there is a Lipschitz two-valued function $v_i \colon D_i \to \mathcal{A}_2((\mathrm{spt}\|\mathbf{C}\|)^\perp)$ satisfying

$$\sup \frac{|v_i|}{r} + \sup |\nabla v_i| \leq \beta,$$
$$\textit{there exists } \Sigma_i \subset D_i \textit{ such that } \mathrm{graph}|_{D_i \setminus \Sigma_i}(v_i) \subset \mathrm{spt}\|V\|.$$

Finally, defining $G := \mathrm{graph}(u) \cup \bigcup_i \mathrm{graph}(v_i)$ we have the estimate

$$\int_{B_\gamma \setminus G} r^2 \mathrm{d}\|V\| + \int_{U^{(1)} \cap B_\gamma} r^2 |\nabla u|^2 \mathrm{d}\mathcal{H}^n + \sum_i \int_{D_i \cap B_\gamma} r^2 |\nabla v_i|^2 \mathrm{d}\mathcal{H}^n \leq C \int_{B_1(0)} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|) \mathrm{d}\|V\|. \tag{3.2.3}$$

Remark 3.2.7. *The set U is the 'good set', i.e. a subset of* $\mathrm{spt}\|\mathbf{C}\|$ *over which V has some form of graphical approximation. We decompose it into two pieces: $U^{(1)}$, the 'really good set', over which V can be parametrised as a smooth single-valued graph; and $U^{(2)}$, the 'pretty good set', over which large parts of V can be parametrised as a two-valued graph.*

Proof. Assume that ε_0 has been chosen at least as small as required for the conclusions of Lemma 3.2.5 to hold. Assume also that $\beta \leq \beta_{\mathbf{C}^{(0)}}$ as defined earlier. In particular this means that we may choose ε_0 sufficiently small that the 2β-conical neighbourhoods of each of the H_i that make up $\mathrm{spt}\|\mathbf{C}\|$ are disjoint.

We consider points $(\xi_i, \zeta_i) \in \mathrm{spt}\|\mathbf{C}\| \cap B_\gamma$ corresponding to a maximal disjoint collection of tori of the form $T_{|\xi_i|, 2c/9}(\zeta_i)$. As established in Lemma 3.2.3, given such a collection we know that $(T_{|\xi_i|, c/2}(\zeta_i))$ is a cover of $B_\gamma(0) \setminus B$ that we can divide into $N(n)$ pairwise disjoint subcollections. We now define U to be the union of $T_{|\xi_i|, c/2}(\zeta_i) \cap \mathrm{spt}\|\mathbf{C}\|$ over all i such that for each of the disks $D^j_{|\xi_i|, 3c/4}(\zeta_i)$,

$j = 1, \dots, 4$, one of the two following cases holds.

(1) There exists $u_{i,j} \in C^2(D^j_{|\xi_i|,3c/4}(\zeta_i); (\mathrm{spt}\|\mathbf{C}\|)^\perp)$ with

$$\begin{aligned} &\mathrm{spt}\|V\| \cap U^j_{|\xi_i|,c/2}(\zeta_i) \subset \mathrm{graph}(u_{i,j}) \subset \mathrm{spt}\|V\|, \text{ and} \\ &\frac{1}{|\xi_i|} \sup_{D^j_{|\xi_i|,3c/4}(\zeta_i)} |u_{i,j}| + \sup_{D^j_{|\xi_i|,3c/4}(\zeta_i)} |\nabla u_{i,j}| \leq \frac{\beta}{2}. \end{aligned}$$

(2) There exists Lipschitz functions $v_{i,j} \colon D^j_{|\xi_i|,3c/4}(\zeta_i) \to \mathcal{A}_2((\mathrm{spt}\|\mathbf{C}\|)^\perp)$ and sets $\Sigma_{i,j} \subset D^j_{|\xi_i|,3c/4}(\zeta_i)$ such that

$$\begin{aligned} \mathcal{H}^n(\Sigma_{i,j}) + &\|V\|(U^j_{|\xi_i|,3c/4}(\zeta_i) \setminus \mathrm{graph}(v_{i,j})) \\ &\leq \frac{C}{|\xi_i|^2} \int_{T_{|\xi_i|,c}(\zeta_i)} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|) \mathrm{d}\|V\|, \\ \frac{1}{|\xi_i|} &\sup_{D^j_{|\xi_i|,3c/4}(\zeta_i)} |v_{i,j}| + \sup_{D^j_{|\xi_i|,3c/4}(\zeta_i)} |\nabla v_{i,j}| \leq \frac{\beta}{2}, \text{ and} \\ &\mathrm{graph}|_{D^j_{|\xi_i|,3c/4}(\zeta_i) \setminus \Sigma_{i,j}} (v_{i,j}) \subset \mathrm{spt}\|V\|. \end{aligned} \tag{3.2.4}$$

We define $U^{(1)}$ to be the union of those $D^j_{|\xi_i|,c/2}(\zeta_i)$ for which alternative (1) holds, and similarly $U^{(2)}$ to be the union of those $D^j_{|\xi_i|,c/2}(\zeta_i)$ for which alternative (2) holds. Moreover we rename the disks making up $U^{(2)}$ as D_j, with the corresponding graph functions and 'bad sets' being denoted v_j and Σ_j respectively. We define $u \in C^2(U^{(1)}; (\mathrm{spt}\|\mathbf{C}\|)^\perp)$ by its restriction to the disks making up $U^{(1)}$. This is indeed well-defined and C^2 by unique continuation of solutions of the minimal surface system. The claimed estimates on u and each v_j follow immediately from the construction.

If $(x, y) \in \mathrm{spt}\|\mathbf{C}\| \cap B_\gamma \cap \partial U$ then $(x, y) \in T_{|\xi_i|,c/2}(\zeta_i)$ for some i with at least one of $D^j_{|\xi_i|,3c/4}(\zeta_i)$ not satisfying either (1) or (2). Hence for this i we must have

$$\int_{T_{|\xi_i|,c}(\zeta_i)} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|) \mathrm{d}\|V\| \geq \frac{\beta^2 |\xi_i|^{n+2}}{C} \tag{3.2.5}$$

for some $C = C(\mathbf{C}^{(0)})$. Indeed if this were not the case, then

$$\frac{1}{|\xi_i|^{n+2}} \int_{T_{|\xi_i|,c}(\zeta_i)} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|) \mathrm{d}\|V\| \leq \frac{\beta^2}{C}.$$

First note that $(\xi_i, \zeta_i) \in B^{k+1}_\tau(0) \times \mathbb{R}^{n-1}$, otherwise by virtue of Lemma 3.2.5, we have that alternative (1) holds on each of the disks $D^j_{|\xi_i|,c/2}(\zeta_i)$. Moreover if C is large enough, then we can guarantee $\operatorname{spt}\|V\|$ and $\operatorname{spt}\|\mathbf{C}\|$ are as close as we like in Hausdorff distance in $T_{|\xi_i|,3c/4}(\zeta_i)$. In particular we can ensure that $\operatorname{dist}(x, \operatorname{spt}\|\mathbf{C}\|) < c(1-\gamma)|\xi_i|/4$ for every $x \in T_{|\xi_i|,3c/4}(\zeta_i) \cap \operatorname{spt}\|V\|$. Applying Lemma 3.2.4 we have that the mass ratios of $U^j_{|\xi_i|,c}(\zeta_i)$ for (ξ_i, ζ_i) in $B_\gamma(0)$ are bounded by $2 + 1/2$. If the mass ratios happen to be bounded by $1 + \varepsilon$, where $\varepsilon > 0$ is as in Allard's regularity theorem (Theorem 2.3.2), then provided C is large enough we can apply Allard's theorem in $U^j_{|\xi_i|,c}(\zeta_i)$ to conclude that in $U^j_{|\xi_i|,3c/4}(\zeta_i)$ alternative (1) holds. Otherwise we know that the mass ratios are between $1+\varepsilon$ and $2+1/2$, so we apply instead Almgren's Lipschitz approximation theorem (Theorem 2.4.3) in $U^j_{|\xi_i|,c}(\zeta_i)$ to conclude that alternative (2) holds in $U^j_{|\xi_i|,3c/4}(\zeta_i)$. The Hausdorff closeness ensures that we have accounted for all of $\operatorname{spt}\|V\|$ in $T_{|\xi_i|,c/2}(\zeta_i)$.

For $(\xi_i, \zeta_i) \in B^{k+1}_\tau(0) \times \mathbb{R}^{n-1}$ as above, we have that $|\xi_i| < \tau \leq (1-\gamma)/10$ and $\|V\|(B_1(0)) \leq \omega_n(2 + 1/2)$. Therefore it follows from the monotonicity formula that

$$\int_{B_{10|\xi_i|}(0,\zeta_i)} r^2 \mathrm{d}\|V\| \leq C|\xi_i|^{n+2} \frac{\|V\|(B_{10|\xi_i|}(0,\zeta_i))}{|\xi_i|^n} \leq C|\xi_i|^{n+2}$$

where $C = C(\mathbf{C}^{(0)}, \gamma)$. The same estimate holds with $\mathbf{C}^{(0)}$ in place of V, and thus, since $|\nabla u| \leq \beta$ on $U^{(1)}$ and $|\nabla v_j| \leq \beta$ on any D_j, we have

$$\int_{U^{(1)} \cap B_{10|\xi_i|}(0,\zeta_i)} r^2 |\nabla u|^2 \mathrm{d}\mathcal{H}^n \leq C\beta^2 |\xi_i|^{n+2}, \text{ and}$$
$$\sum_j \int_{D_j \cap B_{10|\xi_i|}(0,\zeta_i)} r^2 |\nabla v_j|^2 \mathrm{d}\mathcal{H}^n \leq C|\xi_i|^2 \beta^2 \int_{B_{10|\xi_i|}(0,\zeta_i)} \mathrm{d}\|\mathbf{C}\| \leq C\beta^2 |\xi_i|^{n+2}.$$

Hence, for any i such that $T_{|\xi_i|,c/2}(\zeta_i) \cap (\operatorname{spt}\|\mathbf{C}\| \cap B_\gamma \setminus U) \neq \emptyset$, (3.2.5) implies

$$\int_{U^{(1)} \cap B_{10|\xi_i|(0,\zeta_i)}} r^2 |\nabla u|^2 \mathrm{d}\mathcal{H}^n + \sum_j \int_{D_j \cap B_{10|\xi_i|}(0,\zeta_i)} r^2 |\nabla v_j|^2 \mathrm{d}\mathcal{H}^n$$
$$\leq C \int_{T_{|\xi_i|,c}(\zeta_i)} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|) \mathrm{d}\|V\|,$$

where $C = C(\mathbf{C}^{(0)}, \gamma, \beta)$.

We now claim that if I is defined as the set of indices for which we have that

$T_{|\xi_i|,c/2}(\zeta_i) \cap (\operatorname{spt}\|\mathbf{C}\| \cap B_\gamma \setminus U) \neq \emptyset$, then

$$\left\{(x,y) \in U \cap B_\gamma \,\middle|\, \operatorname{dist}((x,y), B_\gamma \cap \partial U) < \frac{|x|}{4}\right\} \subset \bigcup_{i\in I} B_{2|\xi_i|}(0,\zeta_i).$$

Indeed suppose that $(x,y) \in U \cap B_\gamma$ and that $\operatorname{dist}((x,y), B_\gamma \cap \partial U) < |x|/4$. Let $(a,b) \in B_\gamma \cap \partial U$ be such that $|(a,b) - (x,y)| < |x|/4$. Then we have $|a| \leq 5|x|/4$ and $|x| \leq 4|a|/3$. Moreover, for some $i \in I$ we have $(a,b) \in T_{|\xi_i|,c/2}(\zeta_i)$, hence

$$\begin{aligned} |(x,y) - (0,\zeta_i)| &\leq |(x,y) - (a,b)| + |(a,b) - (\xi_i,\zeta_i)| + |\xi_i| \\ &\leq \frac{|x|}{4} + \frac{c(1-\gamma)|\xi_i|}{4} + |\xi_i| \\ &\leq \frac{1}{3}\left(\frac{c(1-\gamma)|\xi_i|}{4} + |\xi_i|\right) + \frac{c(1-\gamma)|\xi_i|}{4} + |\xi_i| \\ &\leq \frac{(5-\gamma)|\xi_i|}{3} \leq 2|\xi_i|, \end{aligned}$$

from which the claim evidently follows. Furthermore we observe that

$$B_{2|\xi_i|}(0,\zeta_i) \cap B_{2|\xi_j|}(0,\zeta_j) = \emptyset \quad \text{implies} \quad T_{|\xi_i|,c}(\zeta_i) \cap T_{|\xi_j|,c}(\zeta_j) = \emptyset.$$

The Vitali covering lemma (see Simon [51]) implies that we may choose a subset $J \subset I$ such that $\{B_{2|\xi_j|}(0,\zeta_j) \mid j \in J\}$ are pairwise disjoint, and

$$\bigcup_{i\in I} B_{2|\xi_i|}(0,\zeta_i) \subset \bigcup_{j\in J} B_{10|\xi_j|}(0,\zeta_j).$$

Defining $A_j := \{(x,y) \in U^{(j)} \cap B_\gamma \mid \operatorname{dist}((x,y), B_\gamma \cap \partial U) < |x|/4\} \subset \operatorname{spt}\|\mathbf{C}\| \cap B_\gamma$

for $j = 1$, 2, we deduce

$$\begin{aligned}
&\int_{A_1} r^2|\nabla u|^2 \mathrm{d}\mathcal{H}^n + \sum_l \int_{A_2\cap D_l} r^2|\nabla v_l|^2 \mathrm{d}\mathcal{H}^n \\
&\quad\leq \sum_{i\in I}\left(\int_{U^{(1)}\cap B_{2|\xi_i|}(0,\zeta_i)} r^2|\nabla u|^2 \mathrm{d}\mathcal{H}^n + \sum_l \int_{U^{(2)}\cap B_{2|\xi_i|}(0,\zeta_i)\cap D_l} r^2|\nabla v_l|^2 \mathrm{d}\mathcal{H}^n\right) \\
&\quad\leq \sum_{j\in J}\left(\int_{U^{(1)}\cap B_{10|\xi_j|}(0,\zeta_j)} r^2|\nabla u|^2 \mathrm{d}\mathcal{H}^n + \sum_l \int_{U^{(2)}\cap B_{10|\xi_j|}(0,\zeta_j)\cap D_l} r^2|\nabla v_l|^2 \mathrm{d}\mathcal{H}^n\right) \\
&\quad\leq \sum_{j\in J} C\int_{T_{|\xi_j|,c}(\zeta_j)} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\| \\
&\quad\leq C\int_{B_1(0)} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\|.
\end{aligned} \tag{3.2.6}$$

We now consider $\{(x, y) \in U \cap B_\gamma \mid \mathrm{dist}((x, y), B_\gamma \cap \partial U) > |x|/4\}$ and seek to prove an analogous estimate. Simon [52] uses interior estimates for solutions to elliptic partial differential equations in his proof. We need to take a different approach since the Lipschitz two-valued approximations are not guaranteed to solve any equation. Instead we make a judicious choice of vector field in the first variation formula. Our choice is motivated by the proof of the well known tilt/height excess estimate for stationary varifolds, and is analogous to the choice made by Krummel-Wickramasekera in [37]. Due to the slightly complex geometry of the open set in which we wish to work, we are forced to use a number of cut-off functions which result in many terms in the computation. However, the essence of the argument is exactly as in the classical estimate. We begin by defining smooth functions

$$\begin{gathered}
\psi\colon [0,\infty) \to [0,1], \quad \psi(t) \equiv 0 \ t \leq 1/8, \quad \psi(t) \equiv 1 \ t \geq 1/4, \quad 0 \leq \psi'(t) \leq 10 \ \forall t, \\
\phi\colon [0,\infty) \to [0,1] \quad \phi(t) \equiv 1 \ t \leq 1, \quad \phi(t) \equiv 0 \ t \geq 2, \quad -2 \leq \phi'(t) \leq 0 \ \forall t, \\
\eta\colon B_1(0) \to [0,1], \quad \eta(x) \equiv 1 \text{ for } |x| \leq \gamma, \quad \eta \equiv 0 \text{ for } (1+\gamma)/2 \leq |x| \leq 1.
\end{gathered}$$

We also consider the four half-planes that make up $\mathrm{spt}\|\mathbf{C}\|$ separately, so without loss of generality let us assume that we are working on H_1. We define coordinates $(x, y) = (\hat{x}, x_{k+1}, y)$ where $\hat{x} \in \mathbb{R}^k$, $x_{k+1} \in \mathbb{R}$ are such that $H_1 = \{(x, y)|\hat{x} = 0,\ x_{k+1} > 0\}$. In particular notice that if p_1 is the orthogonal projection onto the plane containing H_1, then $p_1(x, y) = (0, x_{k+1}, y)$ and $(x, y) - p_1(x, y) = (\hat{x}, 0, 0)$.

Finally we define

$$\xi(x,y) := \operatorname{dist}(p_1(x,y), B_\gamma \cap \partial U),$$

and notice that ξ is Lipschitz with constant 1. With all these in hand we define the following vector field on $\mathbb{R}^{n+k}$.

$$\Phi(x,y) := x_{k+1}^2 \eta^2(x,y) \psi^2 \left(\frac{\xi(x,y)}{|x_{k+1}|} \right) \phi^2 \left(\frac{|\hat{x}|}{\beta |x_{k+1}|} \right) (\hat{x}, 0, 0).$$

Notice that all of the cut-off functions are identically 1 on the β-conical neighbourhood of the set $\{(x,y) \in U \cap B_\gamma | \operatorname{dist}((x,y), B_\gamma \cap \partial U) \geq |x|/4\} \cap H_1$, and identically 0 on the 2β-conical neighbourhood of H_1. Hence the region in which all the cut-off functions are 1 contains all of the pieces of $\operatorname{spt}\|V\|$ that are parametrised as a graph over H_1, and none of the pieces of $\operatorname{spt}\|V\|$ that are graphical over H_i for $i = 2, \ldots, 4$. We want to compute $\operatorname{div}_M \Phi$, so we differentiate to find

$$\begin{aligned} D_i \Phi^j = {} & 2x_{k+1} \eta \psi \phi (\delta_{i,k+1} \eta \psi \phi + x_{k+1} \psi \phi D_i \eta \\ & + x_{k+1} \eta \phi D_i \psi + x_{k+1} \eta \psi D_i \phi) x_j + x_{k+1}^2 \eta^2 \psi^2 \phi^2 \delta_{i,j}, \end{aligned}$$

for $j \leq k$ and $i = 1, \ldots, n+k$. By the chain rule we may further calculate

$$\begin{aligned} D_i \psi &= \psi' \left(\frac{\xi}{|x_{k+1}|} \right) \left(\frac{D_i \xi}{|x_{k+1}|} - \frac{\xi x_{k+1} \delta_{i,k+1}}{|x_{k+1}|^3} \right), \quad \text{and} \\ D_i \phi &= \phi' \left(\frac{|\hat{x}|}{\beta |x_{k+1}|} \right) \left(\frac{(1 - \delta_{i,k+1}) x_j}{\beta |x_{k+1}| |\hat{x}|} - \frac{x_{k+1} |\hat{x}| \delta_{i,k+1}}{\beta |x_{k+1}|^3} \right) \end{aligned}$$

for each $i = 1, \ldots, n+k$. Note in particular that Φ is C^1 and compactly supported in $B_1(0)$. Denoting by p^{ij} the projection to the approximate tangent plane $T_X M$ we have

$$\begin{aligned} \operatorname{div}_M \Phi(x,y) = {} & \sum_{i=1}^{n+k} \sum_{j=1}^{k} p^{ij} x_j 2 x_{k+1} \eta \psi \phi^2 x_{k+1} \eta \psi' \left(\frac{D_i \xi}{|x_{k+1}|} - \frac{\xi x_{k+1} \delta_{i,k+1}}{|x_{k+1}|^3} \right) \\ & + \sum_{i=1}^{n+k} \sum_{j=1}^{k} p^{ij} x_j 2 x_{k+1} \eta \psi \phi^2 \left(x_{k+1} \psi D_i \eta + \eta \psi \delta_{i,k+1} \right) + p^{ij} \delta_{i,j} x_{k+1}^2 \eta^2 \psi^2 \phi^2 \\ & + \sum_{i=1}^{k+1} \sum_{j=1}^{k} p^{ij} x_j 2 x_{k+1}^2 \eta^2 \psi^2 \phi \phi' \left(\frac{(1 - \delta_{i,k+1}) x_i}{\beta |x_{k+1}| |\hat{x}|} - \frac{x_{k+1} |\hat{x}| \delta_{i,k+1}}{\beta |x_{k+1}|^3} \right). \end{aligned}$$

Integrating with respect to $\|V\|$, invoking the first variation formula and applying the Cauchy-Schwarz inequality we find

$$
\begin{aligned}
&\int x_{k+1}^2\eta^2\psi^2\phi^2\sum_{i=1}^{k}p^{ii}\mathrm{d}\|V\|\\
&\quad\le C\int |x_{k+1}||\hat{x}|\eta\phi^2\psi\left(\sum_{i=1}^{n+k}\sum_{j=1}^{k}(p^{ij})^2\right)^{\frac{1}{2}}\\
&\qquad\qquad\cdot\left(\sum_{i=1}^{n+k}x_{k+1}^2\eta^2(\psi')^2\left(\frac{(D_i\xi)^2}{x_{k+1}^2}+\frac{\xi^2x_{k+1}^2\delta_{i,k+1}}{x_{k+1}^6}\right)\right)^{\frac{1}{2}}\mathrm{d}\|V\|\\
&\quad+C\int |x_{k+1}||\hat{x}|\eta\phi^2\psi\left(\sum_{i=1}^{n+k}\sum_{j=1}^{k}(p^{ij})^2\right)^{\frac{1}{2}}\\
&\qquad\qquad\cdot\left(\sum_{i=1}^{n+k}x_{k+1}^2\psi^2(D_i\eta)^2+\eta^2\psi^2\delta_{i,k+1}\right)^{\frac{1}{2}}\mathrm{d}\|V\|\\
&\quad+C\int x_{k+1}^2|\hat{x}|\eta^2\psi^2\phi\left(\sum_{i=1}^{n+k}\sum_{j=1}^{k}(p^{ij})^2\right)^{\frac{1}{2}}\\
&\qquad\qquad\cdot\left(\sum_{i=1}^{n+k}(\phi')^2\left(\frac{(1-\delta_{j,k+1})x_j^2}{\beta^2x_{k+1}^2|\hat{x}|^2}+\frac{x_{k+1}^2|\hat{x}|^2\delta_{i,k+1}}{\beta^2x_{k+1}^6}\right)\right)^{\frac{1}{2}}\mathrm{d}\|V\|.
\end{aligned}
$$

We observe that since p^{ij} is the matrix of a projection map, we have

$$
\sum_{i=1}^{n+k}\sum_{j=1}^{k}(p^{ij})^2=\sum_{j=1}^{k}p^{jj}.
$$

Hence using the weighted Young's inequality we find

$$
\begin{aligned}
&\int x_{k+1}^2\eta^2\psi^2\phi^2\sum_{j=1}^{k}p^{jj}\mathrm{d}\|V\|\\
&\quad\le C\int\phi^2|\hat{x}|^2\left(x_{k+1}^2\eta^2(\psi')^6\left(\frac{|D\xi|^2}{x_{k+1}^2}+\frac{\xi^2x_{k+1}^2}{x_{k+1}^6}\right)+x_{k+1}^2\psi^2|D\eta|^2+\eta^2\psi^2\right)\mathrm{d}\|V\|\\
&\qquad+C\int x_{k+1}^2\eta^2\psi^2|\hat{x}|^2(\phi')^2\left(\frac{1}{\beta^2x_{k+1}^2}+\frac{x_{k+1}^2|\hat{x}|^2}{\beta^2x_{k+1}^6}\right)\mathrm{d}\|V\|.
\end{aligned}
$$

We observe that $\psi'=0$ unless $|x_{k+1}|/8\le\xi\le|x_{k+1}|/4$ and $\phi'=0$ unless

$\beta|x_{k+1}| \leq |\hat{x}| \leq 2\beta|x_{k+1}|$, hence we arrive at

$$\int x_{k+1}^2 \eta^2 \psi^2 \phi^2 \sum_{j=1}^{k} p^{jj} \mathrm{d}\|V\| \leq C \int_{C(H_1, 2\beta)} |\hat{x}|^2 \mathrm{d}\|V\|.$$

Now if (p^{ij}) denotes the matrix of the projection to $T_X M$ wherever it exists, and (ε^{ij}) denotes the matrix of the projection to $\mathbb{R}^n \times \{0\}^k$, then

$$\begin{aligned}
\sum_{j=1}^{k} p^{jj} = n - \sum_{j=k+1}^{n+k} p^{jj} &= \frac{1}{2} \sum_{i,j=1}^{n+k} (p^{ij})^2 + (\varepsilon^{ij})^2 - 2p^{ij}\varepsilon^{ij} \\
&= \frac{1}{2} \sum_{i,j=1}^{n+k} (p^{ij} - \varepsilon^{ij})^2 = \frac{1}{2} |p_{T_X M} - p|^2,
\end{aligned}$$

and

$$\sum_{i=1}^{k} \sum_{j=1}^{n+k} (p^{ij})^2 = \sum_{i=1}^{k} p^{ii} = \frac{1}{2} |p_{T_X M} - p|^2.$$

Moreover we have that

$$\nabla^M x_j = p_{T_X M}(D x_j) = p_{T_X M}(e_j) = \sum_{i=1}^{n+k} p^{ij} e_i.$$

So therefore

$$|\nabla^M x_j|^2 = \left| \sum_{i=1}^{n+k} p^{ij} e_i \right|^2 = \sum_{i=1}^{n+k} (p^{ij})^2 = p^{jj}.$$

Hence we have

$$\frac{1}{2} |p_{T_X M} - p|^2 = \sum_{i=1}^{k} p^{ii} = \sum_{i=1}^{k} |\nabla^M x_i|^2.$$

Let $\Omega_1 := \{(x,y) \in B_1 \,|\, \mathrm{dist}(p_1(x,y), B_\gamma \cap \partial U) \geq |x_{k+1}|/4\} \cap C_{H_1}(U \cap B_\gamma \cap H_1, \beta)$, then we have

$$\begin{aligned}
\int_{\Omega_1} x_{k+1}^2 |\nabla^M \hat{x}|^2 \mathrm{d}\|V\| &\leq C \int_{C(H_1, 2\beta) \cap B_1} |\hat{x}|^2 \mathrm{d}\|V\| \\
&\leq C \int_{B_1} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|) \mathrm{d}\|V\|.
\end{aligned}$$

Defining $W_j := \{(x,y) \in U^{(j)} \cap B_\gamma \,|\, \mathrm{dist}((x,y), B_\gamma \cap \partial U) \geq |x|/4\} \subset \mathrm{spt}\|\mathbf{C}\| \cap B_\gamma$ for $j = 1$, 2, we deduce

$$\int_{W_1 \cap H_1} r^2 |\nabla u|^2 \mathrm{d}\mathcal{H}^n \leq \int_{\Omega_1} x_{k+1}^2 |\nabla^M \hat{x}|^2 \mathrm{d}\|V\| \leq C \int_{B_1} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}\|) \mathrm{d}\|V\|$$

and similarly

$$\begin{aligned}
&\sum_j \int_{W_2\cap H_1\cap D_j} r^2|\nabla v_j|^2 \mathrm{d}\mathcal{H}^n \\
&= \sum_j \left(\int_{W_2\cap H_1\cap D_j\setminus\Sigma_j} r^2|\nabla v_j|^2\mathrm{d}\mathcal{H}^n + \int_{W_2\cap H_1\cap\Sigma_j} r^2|\nabla v_j|^2\mathrm{d}\mathcal{H}^n \right) \\
&\leq C\int_{\Omega_1} x_{k+1}^2|\nabla^M\hat{x}|^2\mathrm{d}\|V\| + \sum_j C|\xi_j|^2\mathcal{H}^n(\Sigma_j\cap H_1) \\
&\leq C\int_{C(H_1,2\beta)\cap B_1} |\hat{x}|^2\mathrm{d}\|V\| + C\sum_j\int_{T_{|\xi_j|,c}(\zeta_j)} \mathrm{dist}^2(X,\mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\| \\
&\leq C\int_{B_1} \mathrm{dist}^2(X,\mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\|.
\end{aligned}$$

Of course the same estimate holds with any H_i in place of H_1, and so we have

$$\int_{W_1} r^2|\nabla u|^2\mathrm{d}\mathcal{H}^n + \sum_j\int_{W_2\cap D_j} r^2|\nabla v_j|^2\mathrm{d}\mathcal{H}^n \leq C\int_{B_1}\mathrm{dist}^2(X,\mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\|. \tag{3.2.7}$$

Finally we consider again i such that $T_{|\xi_i|,c/2}(\zeta_i)\cap(\mathrm{spt}\|\mathbf{C}\|\cap B_\gamma\setminus U)\neq\emptyset$. Recall we denote by I the collection of such i. Then arguing as before we find

$$\int_{B_{10|\xi_i|}(0,\zeta_i)} r^2\mathrm{d}\|V\| \leq C\int_{T_{|\xi_i|,c}(\zeta_i)}\mathrm{dist}^2(X,\mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\|,$$

for some $C = C(\mathbf{C}^{(0)},\beta)$ and for each $i\in I$. By definition of U, the union of such tori contains all the pieces of $\mathrm{spt}\|V\|$ which are not parametrised as graphs, and each torus is contained in the corresponding ball $B_{2|\xi_i|}(0,\zeta_i)$. Hence, employing the Vitali covering lemma as before, we find a subset $J\subset I$ such that the collection $\{B_{2|\xi_j|}(0,\zeta_j)\mid j\in J\}$ is disjoint but with

$$\bigcup_{i\in I} B_{2|\xi_i|}(0,\zeta_i)\subset\bigcup_{j\in J} B_{10|\xi_j|}(0,\zeta_j). \tag{3.2.8}$$

Then since $\mathrm{spt}\|V\|\setminus G$, where $G=\mathrm{graph}(u)\cup\bigcup_l\mathrm{graph}(v_l)$, is contained in the union of $\mathrm{spt}\|V\|\cap T_{|\xi_i|,c}(\zeta_i)$ over all $i\in I$, and the union of $U_l\cap(\mathrm{spt}\|V\|\setminus\mathrm{graph}(v_l))$ over l, where U_l is the open ball centred on $\mathrm{spt}\|\mathbf{C}\|$ whose intersection with $\mathrm{spt}\|\mathbf{C}\|$

is precisely D_l, combining (3.2.8), and (3.2.4) we find

$$\begin{aligned}\int_{B_\gamma\setminus G} r^2 \mathrm{d}\|V\| &\le \sum_i \int_{B_\gamma\cap B_{2|\xi_i|}(0,\zeta_i)} r^2\mathrm{d}\|V\| + \sum_l \int_{B_\gamma\cap U_l\setminus \mathrm{graph}(v_l)} r^2 \mathrm{d}\|V\| \\ &\le \sum_j \int_{B_{10|\xi_j|}(0,\zeta_j)} r^2\mathrm{d}\|V\| + \sum_l C|\xi_l|^2\|V\|(U_l\setminus \mathrm{graph}(v_l)) \\ &\le \sum_j C\int_{T_{|\xi_j|,c}(\zeta_j)} \mathrm{dist}^2(X,\mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\| \\ &\quad + \sum_l C\int_{T_{|\xi_l|,c}(\zeta_l)} \mathrm{dist}^2(X,\mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\| \\ &\le C\int_{B_1} \mathrm{dist}^2(X,\mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\|. \end{aligned} \tag{3.2.9}$$

Combining (3.2.6), (3.2.7) and (3.2.9) the conclusions and (3.2.3) follow. □

3.3 L^2-estimates

In this section we use the graphical estimates of the previous section to prove the following L^2-estimates. These are analogues of Simon's main L^2-estimates (cf. [52, Theorem 3.1]).

Theorem 3.3.1 (Main L^2-estimates)**.** *Let $\mathbf{C}^{(0)} \in \mathcal{C}$ and γ, $\tau \in (0,1)$. There exist $\varepsilon_0 = \varepsilon(\mathbf{C}^{(0)},\gamma,\tau)$, $\beta_0 = \beta_0(\mathbf{C}^{(0)}) \in (0,1)$ such that the following holds. If V, $\mathbf{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_0$, $\delta_A = 1/8$ and u is as in Lemma 3.2.6 with $\beta = \beta_0$, $\tau = \tau$, then for any $Z = (\xi,\eta) \in B_{3/4}\cap \mathrm{spt}\|V\|$ with $\Theta(\|V\|,Z) \ge \Theta(\|\mathbf{C}^{(0)}\|,0) = 2$, we have*

$$|\xi|^2 + \int_{B_\gamma} \sum_{j=1}^{n-1} |e^\perp_{k+1+j}|^2 \mathrm{d}\|V\| + \int_{B_\gamma} \frac{d^2}{|X-Z|^{n-1/4}}\mathrm{d}\|V\| \le C\int_{B_1} d^2 \mathrm{d}\|V\|, \tag{3.3.1a}$$

$$\int_{\{X\in \mathbf{C}\cap B_\gamma \,|\, |x|>\tau\}} \frac{|u(X)-\xi^\perp(X)|^2}{|X-Z|^{n+7/4}} \mathrm{d}\mathcal{H}^n \le C\int_{B_1} d_Z^2 \mathrm{d}\|V\|, \tag{3.3.1b}$$

where $\xi^\perp(x,y)$ is the orthogonal projection of $(\xi,0)$ onto $(T_X\mathbf{C})^\perp$, C depends only on γ and $\mathbf{C}^{(0)}$ and we used the notation $d(X) := \mathrm{dist}(X,\mathrm{spt}\|\mathbf{C}\|)$ and $d_Z(X) := \mathrm{dist}(X,\mathrm{spt}\|T_{Z\#}\mathbf{C}\|)$.

We will break down the proof of the above statement into multiple intermediate lemmas, but before doing so we state the following important corollary to Theorem 3.3.1. It implies that, provided there is an abundance of good density points, the height excess can't concentrate in small cylindrical neighbourhoods of B.

Corollary 3.3.2. *Let $\mathbf{C}^{(0)} \in \mathcal{C}$, and $\delta \in (0, 1/8)$. There is $\varepsilon_0 = \varepsilon_0(\mathbf{C}^{(0)})$ such that if V, $\mathbf{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \min\{\varepsilon_0, \delta\}$ and $\delta_A = 1/8$, and*

$$\{0\}^{k+1} \times \bar{B}^{n-1}_{1/2} \subset B_{2\delta}(\{X \mid \Theta(\|V\|, X) \geq \Theta(\|\mathbf{C}^{(0)}\|, 0) = 2\}), \tag{3.3.2}$$

then

$$\int_{B_{1/2}} \frac{d^2}{r_\delta^{1/2}} \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|, \tag{3.3.3}$$

where $C = C(\mathbf{C}^{(0)})$, $r_\delta := \max\{|x|, \delta\}$.

Remark 3.3.3. *Notice that* (3.3.3) *implies*

$$\int_{(B^{k+1}_\delta \times \mathbb{R}^{n-1}) \cap B_{1/2}} d^2 \mathrm{d}\|V\| \leq C\delta^{1/2} \int_{B_1} d^2 \mathrm{d}\|V\|,$$

with C independent of δ. Therefore the part of $\operatorname{spt}\|V\| \cap B_{1/2}$ close to B contributes little to $\int_{B_1} d^2 \mathrm{d}\|V\|$ if the hypotheses of Corollary 3.3.2 hold with δ small enough (which depends only on $\mathbf{C}^{(0)}$).

Proof of Corollary 3.3.2. Let $z \in B^{n-1}_{1/2}(0)$. Then by (3.3.2) we know that there exists Z with $\Theta(\|V\|, Z) \geq 2$ and $|Z - (0, z)| < 2\delta$. From (3.3.1a) of Theorem 3.3.1 we know

$$\frac{1}{\rho^{n-1/4}} \int_{B_\rho(0,z)} d^2 \mathrm{d}\|V\| \leq C \int_{B_{1/2}} \frac{d^2}{|X - Z|^{n-1/4}} \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|,$$

for any $\rho \in (2\delta, 1/4)$. We can cover $B_{1/2} \times (B^{k+1}_{\rho/2} \times \mathbb{R}^{n-1})$ by $N \leq C(n, k)\rho^{-(n-1)}$ balls $B_\rho(0, z_j)$ with $|z_j| \leq 1/2$ for each j, and such that $\{B_\rho(0, z_j)\}$ splits into at most $C(n, k)$ pairwise disjoint subcollections. Therefore denoting by J the index set of the collection $\{B_\rho(0, z_j)\}$ it follows that for all $\rho \in (2\delta, 1/4)$

$$\frac{1}{\rho^{3/4}} \int_{B_{1/2} \cap (B^{k+1}_{\rho/2} \times \mathbb{R}^{n-1})} d^2 \mathrm{d}\|V\| \leq \sum_{j \in J} \frac{1}{\rho^{3/4}} \int_{B_\rho(0,z_j)} d^2 \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|.$$

Multiplying by $\rho^{-3/4}$ we have

$$\frac{1}{\rho^{3/2}}\int_{B_{1/2}\cap(B^{k+1}_{\rho/2}\times\mathbb{R}^{n-1})} d^2 \mathrm{d}\|V\| \leq \frac{C}{\rho^{3/4}}\int_{B_1} d^2\mathrm{d}\|V\|.$$

Integrating this over $(2\delta, 1/4)$ with respect to ρ we find

$$\int_{2\delta}^{1/4}\frac{1}{\rho^{3/2}}\int_{B_{1/2}\cap(B^{k+1}_{\rho/2}\times\mathbb{R}^{n-1})} d^2 \mathrm{d}\|V\|\mathrm{d}\rho \leq C\left(\frac{1}{4^{1/4}}-(2\delta)^{1/4}\right)\int_{B_1} d^2\mathrm{d}\|V\|.$$

The left hand side equals

$$\begin{aligned}\int_{B_{1/2}} d^2\int_{2\delta}^{1/4}\frac{1}{\rho^{3/2}}\chi_{B^{k+1}_{\rho/2}\times\mathbb{R}^{n-1}}\mathrm{d}\rho\mathrm{d}\|V\| &= \int_{B_{1/2}} d^2\int_{2r_\delta}^{1/8}\frac{1}{\rho^{3/2}}\mathrm{d}\rho\mathrm{d}\|V\| \\ &= 2\int_{B_{1/2}} d^2\left(\frac{1}{(2r_\delta)^{1/2}}-\frac{1}{4^{1/2}}\right)\mathrm{d}\|V\|,\end{aligned}$$

where $r_\delta = \max\{|x|,\delta\}$. Rearranging we have exactly the desired estimate

$$\int_{B_{1/2}}\frac{d^2}{r_\delta^{1/2}}\mathrm{d}\|V\| \leq C\int_{B_1} d^2\mathrm{d}\|V\|.$$

□

We now embark on the proof of Theorem 3.3.1. The first step is to prove an analogue of [52, Lemma 3.4]. The proof is completely analogous to Simon's, the only difference being the application of Lemma 3.2.6 as we have to deal with regions of the support of V that have been parametrised as a two-valued graph. This does not substantially alter the argument however.

Lemma 3.3.4. *Let $\mathbf{C}^{(0)} \in \mathcal{C}$ and α, $\gamma \in (0,1)$. There exists $\varepsilon_0 = \varepsilon_0(\mathbf{C}^{(0)},\gamma)$ and $\beta_0 = \beta_0(\mathbf{C}^{(0)}) > 0$ such that if V, $\mathbf{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_0$ and $\delta_A = 1/4$, and if $\Theta(\|V\|,0) \geq \Theta(\|\mathbf{C}^{(0)}\|,0) = 2$, then*

$$\begin{aligned}\int_{B_\gamma}\frac{|X^\perp|^2}{R^{n+2}}\mathrm{d}\|V\| + \int_{B_\gamma}\sum_{j=1}^{m}|e^\perp_{l+k+j}|^2\mathrm{d}\|V\| + \int_{B_\gamma}\frac{d^2}{R^{n+2-\alpha}}\mathrm{d}\|V\| \\ \leq C\int_{B_1}\mathrm{dist}^2(X,\mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\|,\end{aligned} \tag{3.3.4}$$

where $C = C(\mathbf{C}^{(0)},\alpha,\gamma)$, $R(x,y) = \sqrt{|x|^2+|y|^2}$.

Proof. By the monotonicity formula, we know that the mass ratios satisfy

$$\frac{d}{d\rho}\frac{\|V\|(B_\rho)}{\rho^n} = \frac{d}{d\rho}\int_{B_\rho}\frac{|X^\perp|^2}{R^{n+2}}\mathrm{d}\|V\|.$$

Since $\Theta(\|V\|,0) \geq \Theta(\|\mathbf{C}^{(0)}\|,0)$ we find

$$\begin{aligned}
&\frac{d}{d\rho}(\|V\|(B_\rho) - \|\mathbf{C}^{(0)}\|(B_\rho)) \\
&\quad = n\rho^{n-1}\left(\frac{\|V\|(B_\rho)}{\rho^n} - \frac{\|\mathbf{C}^{(0)}\|(B_\rho)}{\rho^n}\right) + \rho^n\frac{d}{d\rho}\left(\frac{\|V\|(B_\rho)}{\rho^n} - \frac{\|\mathbf{C}^{(0)}\|(B_\rho)}{\rho^n}\right) \\
&\quad \geq n\rho^{n-1}\int_{B_\rho}\frac{|X^\perp|^2}{R^{n+2}}\mathrm{d}\|V\|,
\end{aligned}$$

for almost every $\rho \in (0,1]$. Indeed the second term is non-negative by the monotonicity formula and the fact that $\mathbf{C}^{(0)}$ is a cone. Moreover, in the first term we have $\rho^{-n}\|\mathbf{C}^{(0)}\|(B_\rho) = \omega_n\Theta(\|\mathbf{C}^{(0)}\|,0) \leq \omega_n\Theta(\|V\|,0)$, and so we recover precisely the remainder term from the usual monotonicity formula.

Now let $\psi\colon \mathbb{R} \to [0,1]$ be C^2 with $\psi(t) \equiv 1$ for $t < (1+\gamma)/2$, and $\psi(t) \equiv 0$ for $t > (3+\gamma)/4$, with $\psi'(t) \leq 0$. We multiply both sides by $\psi^2(\rho)$ and integrate on $[0,1]$ to get

$$\begin{aligned}
\int_0^1 n\psi^2(\tau)\tau^{n-1}\int_{B_\tau}\frac{|X^\perp|^2}{R^{n+2}}\mathrm{d}\|V\|\mathrm{d}\tau &\leq \int_0^1\psi^2(\tau)\frac{d}{d\tau}\left(\|V\|(B_\tau) - \|\mathbf{C}^{(0)}\|(B_\tau)\right)\mathrm{d}\tau \\
&= \int_{B_1}\psi^2(R)\mathrm{d}\|V\| - \int_{B_1}\psi^2(R)\mathrm{d}\|\mathbf{C}^{(0)}\|.
\end{aligned}$$

Estimating the left hand side from below we find

$$\begin{aligned}
\int_{B_1}\psi^2(R)\mathrm{d}\|V\| - \int_{B_1}\psi^2(R)\mathrm{d}\|\mathbf{C}^{(0)}\| &\geq \int_\gamma^{(\gamma+1)/2} n\tau^{n-1}\int_{B_\tau}\frac{|X^\perp|^2}{R^{n+2}}\mathrm{d}\|V\|\mathrm{d}\tau \\
&\geq \frac{\gamma^{n-1}(1-\gamma)}{2}\int_{B_\gamma}\frac{|X^\perp|^2}{R^{n+2}}\mathrm{d}\|V\|.
\end{aligned} \tag{3.3.5}$$

We now use [52, Equation 2.5], which reads

$$\begin{aligned}
&\int\left(1 + \frac{1}{2}\sum_{i=1}^{n-1}|e_{k+1+i}^\perp|^2\right)\psi^2(R)\mathrm{d}\|V\| \\
&\quad \leq \int\left(-\sum_{i,j=1}^{k+1}g^{ij}x^iD_{x^i}\psi^2 + 2|(x,0)^\perp|^2|D_y\psi|^2\right)\mathrm{d}\|V\|.
\end{aligned} \tag{3.3.6}$$

This is a simple consequence of stationarity, and follows from making a particular choice of vector field in the first variation formula. We let $\psi = \psi(R)$, then $D_{x^i}\psi = x^i\psi'/R$, and $D_{y^i}\psi = y^i\psi'/R$, so

$$\begin{aligned}\int_{B_1}\left(1+\frac{1}{2}\sum_{i=1}^{n-1}|e_{k+1+i}^{\perp}|^2\right)\psi^2(R)\mathrm{d}\|V\| \\ \leq \int_{B_1} -2\psi(R)\psi'(R)R^{-1}\sum_{i,j=1}^{k+1} g^{ij}x^ix^j + 2|(x,0)^{\perp}|^2(\psi'(R))^2\mathrm{d}\|V\|.\end{aligned} \tag{3.3.7}$$

Now

$$\sum_{i,j=1}^{k+1} g^{ij}x^ix^j = |(x,0)^T|^2 = r^2 - |(x,0)^{\perp}|^2,$$

so from (3.3.7) we have

$$\begin{aligned}\int_{B_1}\left(1+\frac{1}{2}\sum_{i=1}^{n-1}|e_{k+1+i}^{\perp}|^2\right)\psi^2(R)\mathrm{d}\|V\| \\ \leq \int -2r^2R^{-1}\psi(R)\psi'(R) + 2|(x,0)^{\perp}|^2(R^{-1}\psi(R)\psi'(R) + (\psi'(R))^2)\mathrm{d}\|V\| \\ \leq C\int_{B_{\tilde{\gamma}}} |(x,0)^{\perp}|^2\mathrm{d}\|V\| - 2\int_{B_1} r^2R^{-1}\psi(R)\psi'(R)\mathrm{d}\|V\|,\end{aligned} \tag{3.3.8}$$

where $C = C(\gamma)$ and $\tilde{\gamma} = (3+\gamma)/4$. Assume that ε_0 is small enough that we may apply Lemma 3.2.6 with $\beta = \min\{\beta_0, \beta_{\mathbf{C}^{(0)}}\}$, $\gamma = \tilde{\gamma}$ and $\tau = (1-\tilde{\gamma})/10$. Notice that if $P_{(x,y)}$ and $Q_{(x',y)}$ denote the orthogonal projections onto $(T_{(x,y)}M)^{\perp}$ and $(T_{(x',y)}\mathbf{C})^{\perp}$ respectively, then

$$\begin{aligned}u(x',y) + (P_{(x,y)} - Q_{(x',y)})(x,0) &= P_{(x,y)}(x,0) + Q_{(x',y)}(x,y) - Q_{(x',y)}(x,0) \\ &= (x,0)^{\perp} + Q_{(x',y)}(0,y) \\ &= (x,0)^{\perp},\end{aligned}$$

for any $(x,y) = (x',y) + u(x',y) \in \operatorname{graph}(u) \subset \operatorname{spt}\|V\|$. Since $|\nabla u| \leq \beta \leq 1$ we also have

$$|P_{(x,y)} - Q_{(x',y)}| \leq C(n,k)|\nabla u(x',y)|,$$

for such (x,y). The same estimates hold for $\mathcal{H}^n$-almost every (x,y) such that

$(x,y) = (x',y) + v_j(x',y) \in \mathrm{graph}(v_j) \subset \mathrm{spt}\|V\|$ for some j. Hence

$$\begin{aligned}\int_{B_{\tilde{\gamma}}} |(x,0)^{\perp}|^2 \mathrm{d}\|V\| - 2\int_{B_1} r^2 R^{-1}\psi(R)\psi'(R)\mathrm{d}\|V\| \\ \leq C\int_{U^{(1)}\cap B_{\tilde{\gamma}}} |u|^2 + r^2|\nabla u|^2 \mathrm{d}\mathcal{H}^n + \sum_j C\int_{D_j\cap B_{\tilde{\gamma}}} |v_j|^2 + r^2|\nabla v_j|^2\mathrm{d}\mathcal{H}^n \\ + C\int_{B_{\tilde{\gamma}}\setminus G} r^2 \mathrm{d}\|V\| - 2\int_{B_1\cap G} r^2R^{-1}\psi(R)\psi'(R)\mathrm{d}\|V\|,\end{aligned} \tag{3.3.9}$$

where the part of the final integral over the non-graphical part has been absorbed into the penultimate integral, using the fact that we have a lower bound on R where $\psi' \neq 0$. Now $\mathbf{C}_0$ consists of two lines crossing at the origin, so for $\phi \in C^\infty(0,\infty)$ with $\phi \equiv$ const in a neighbourhood of the origin, and $\phi \equiv 0$ outside a bounded set, we have

$$\begin{aligned}\int \phi^2(r)\mathrm{d}\|\mathbf{C}_0\| = 4\int_0^\infty \phi^2(r)\mathrm{d}r = 4\left[\phi^2(r)r\right]_0^\infty - 8\int_0^\infty r\phi(r)\phi'(r)\mathrm{d}r \\ = -2\int r\phi(r)\phi'(r)\mathrm{d}\|\mathbf{C}_0\|.\end{aligned}$$

We use this observation with $\phi(r) = \psi(R)$ for y fixed and $R = \sqrt{r^2+|y|^2}$. Then $\phi'(r) = r\psi'(R)/R$, so

$$\int \psi^2(R)\mathrm{d}\|\mathbf{C}_0\| = -2\int r^2R^{-1}\psi(R)\psi'(R)\mathrm{d}\|\mathbf{C}_0\|.$$

Integrating with respect to y we recover

$$\int_{B_1} \psi^2(R)\mathrm{d}\|\mathbf{C}\| = -2\int_{B_1} r^2R^{-1}\psi(R)\psi'(R)\mathrm{d}\|\mathbf{C}\|. \tag{3.3.10}$$

From the area formula we know that

$$\int_{\mathrm{graph}(u)\cap B_1} r^2R^{-1}\psi(R)\psi'(R)\mathrm{d}\mathcal{H}^n = \int_{U^{(1)}\cap B_1} r_u^2R_u^{-1}\psi(R_u)\psi'(R_u)\sqrt{g}\mathrm{d}\mathcal{H}^n,$$

where $\sqrt{g} = 1+E$, with $0 \leq E \leq C|\nabla u|^2 \leq C(r^{-2}|u|^2 + |\nabla u|^2)$, and where $r_u^2 = |x|^2 + |u(x,y)|^2$, $R_u^2 = |x|^2+|u(x,y)|^2+|y|^2$. Since ψ is monotone decreasing, and $R \leq R_u$ it follows

$$-2\int_{U^{(1)}\cap B_1} r_u^2R_u^{-1}\psi(R_u)\psi'(R_u)\sqrt{g}\mathrm{d}\mathcal{H}^n$$

$$
\begin{aligned}
&\le -2\int_{U^{(1)}\cap B_1}(r^2+|u|^2)R^{-1}\psi(R)\psi'(R_u)\sqrt{g}\mathrm{d}\mathcal{H}^n\\
&= -2\int_{U^{(1)}\cap B_1} r^2R^{-1}\psi(R)\psi'(R)\mathrm{d}\mathcal{H}^n - 2\int_{U^{(1)}\cap B_1} r^2R^{-1}\psi(R)\psi'(R)E\mathrm{d}\mathcal{H}^n\\
&\quad - 2\int_{U^{(1)}\cap B_1}(r^2+|u|^2)R^{-1}\psi(R)(\psi'(R_u)-\psi'(R))\sqrt{g}\mathrm{d}\mathcal{H}^n\\
&\quad - 2\int_{U^{(1)}\cap B_1} r^2R^{-1}\psi(R)\psi'(R)\mathrm{d}\mathcal{H}^n.
\end{aligned}
$$

Therefore, using the fact that $|\psi'(R_u)-\psi'(R)|\le C|u|^2$, $|u|^2\le r^2$ and $r\le R$, and the bound on E we have

$$
\begin{aligned}
&-2\int_{U^{(1)}\cap B_1} r_u^2R_u^{-1}\psi(R_u)\psi'(R_u)\sqrt{g}\mathrm{d}\mathcal{H}^n\\
&\quad\le -2\int_{U^{(1)}\cap B_1} r^2R^{-1}\psi(R)\psi'(R)\mathrm{d}\mathcal{H}^n + C\int_{U^{(1)}\cap B_{\tilde\gamma}}|u|^2+r^2|\nabla u|^2\mathrm{d}\mathcal{H}^n.
\end{aligned}
\tag{3.3.11}
$$

Arguing completely analogously, it follows that

$$
\begin{aligned}
&-2\int_{D_j\cap B_1} r_{v_j}^2R_{v_j}^{-1}\psi(R_{v_j})\psi'(R_{v_j})\sqrt{g_j}\mathrm{d}\mathcal{H}^n\\
&\quad\le -2\int_{D_j\cap B_1} r^2R^{-1}\psi(R)\psi'(R)\mathrm{d}\mathcal{H}^n + C\int_{D_j\cap B_{\tilde\gamma}}|v_j|^2+r^2|\nabla v_j|^2\mathrm{d}\mathcal{H}^n.
\end{aligned}
\tag{3.3.12}
$$

Hence from (3.3.10), (3.3.9), (3.3.11) and (3.3.12) it follows that

$$
\begin{aligned}
&\frac{1}{2}\int_{B_1}\left(\sum_{j=1}^{n-1}|e_{k+1+j}^{\perp}|^2\right)\psi^2(R)\mathrm{d}\|V\| + \int_{B_1}\psi^2(R)\mathrm{d}\|V\| - \int_{B_1}\psi^2(R)\mathrm{d}\|\mathbf{C}\|\\
&\quad\le C\int_{U^{(1)}\cap B_{\tilde\gamma}}|u|^2+r^2|\nabla u|^2\mathrm{d}\mathcal{H}^n + \sum_j C\int_{D_j\cap B_{\tilde\gamma}}|v_j|^2+r^2|\nabla v_j|^2\mathrm{d}\mathcal{H}^n\\
&\qquad + C\int_{B_{\tilde\gamma}\setminus G} r^2\mathrm{d}\|V\|.
\end{aligned}
$$

Applying the estimate (3.2.3) from Lemma 3.2.6 and using (3.3.5) we find

$$
\int_{B_\gamma}\frac{|X^\perp|^2}{R^{n+2}}\mathrm{d}\|V\| + \int_{B_\gamma}\sum_{j=1}^{n-1}|e_{k+1+j}^{\perp}|^2\mathrm{d}\|V\| \le C\int_{B_1}\mathrm{dist}^2(X,\mathrm{spt}\|\mathbf{C}\|)\mathrm{d}\|V\|. \tag{3.3.13}
$$

Next we establish the bound on

$$\int_{B_\gamma} \frac{\operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}\|)}{R^{n+2-\alpha}} \mathrm{d}\|V\|.$$

Note that $d\colon \mathbb{R}^{n+k} \to \mathbb{R}$, $X \mapsto \operatorname{dist}(X, \operatorname{spt}\|\mathbf{C}\|)$ is homogeneous degree 1 with Lipschitz constant 1. Moreover, if $X = (x, y)$ then d is independent of the y variable, and d^2 is C^1 in the x variable in the region

$$K_{\varepsilon_0} := \left\{(x, y) \in (\mathbb{R}^{k+1} \setminus \{0\}) \times \mathbb{R}^{n-1} \mid \operatorname{dist}((x, y), \operatorname{spt}\|\mathbf{C}\|) \leq \varepsilon_0 |x|\right\}.$$

Hence it easily follows that we may construct $\tilde{d}\colon \mathbb{R}^{n+k} \to \mathbb{R}$ with Lipschitz constant C, that is homogeneous degree 1, with $\tilde{d} \equiv d$ on K_{ε_0}, such that $\tilde{d}^2$ is C^1 everywhere and

$$\frac{1}{C} d(X) \leq \tilde{d}(X) \leq C d(X),$$

where $C = C(\mathbf{C}^{(0)})$. We now define

$$\Phi(X) := \frac{\zeta^2}{R^{n-\alpha}} \frac{\tilde{d}^2}{R^2} X,$$

where $\zeta \in C^\infty(\mathbb{R}^{n+k})$ satisfies $\zeta \equiv 1$ on $B_{(1+\gamma)/2}$, $\zeta \equiv 0$ outside of B_1 and $|\nabla^{\mathbb{R}^{n+k}} \zeta| \leq C$, where $C = C(\gamma)$. Since $\tilde{d}^2/R^2$ is C^1 and homogeneous degree 0 (i.e. radially constant) away from B, it follows that

$$X \cdot \nabla^{\mathbb{R}^{n+k}} \frac{\tilde{d}^2}{R^2} = \sum_{j=1}^{n+k} X^j D_j \frac{\tilde{d}^2}{R^2} = 0.$$

Furthermore, since $\operatorname{Lip}(\tilde{d}) \leq C$ and $\tilde{d} \leq CR$ we have

$$\left|\nabla^{\mathbb{R}^{n+k}} (R^{-1}\tilde{d})\right| \leq \frac{2C}{R}.$$

Also, $\operatorname{div}_M X \equiv n$, $|\nabla^M R| \leq |\nabla^{\mathbb{R}^{n+k}} R| \leq 1$, so

$$\begin{aligned} \operatorname{div}_M \Phi(X) = {} & \frac{\zeta^2}{R^{n-\alpha}} \frac{\tilde{d}^2}{R^2} n + \frac{1}{R^{n-\alpha}} \frac{\tilde{d}^2}{R^2} 2\zeta \nabla^M \zeta \cdot X + 2 \frac{\zeta^2}{R^{n-\alpha}} \frac{\tilde{d}}{R} \nabla^M \frac{\tilde{d}}{R} \cdot X \\ & + \zeta^2 \frac{\tilde{d}^2}{R^2} \frac{(\alpha - n)}{R^{n+1-\alpha}} \nabla^M R \cdot X \end{aligned}$$

$$= \frac{1}{R^{n-\alpha}} \sum_{i,j=1}^{n+k} X^i \left(2\zeta^2 \frac{\tilde{d}}{R} (g^{ij} - \delta_{ij}) D_j \frac{\tilde{d}}{R} + 2\zeta \frac{\tilde{d}^2}{R^2} g^{ij} D_j \zeta \right)$$
$$+ \frac{\zeta^2}{R^{n-\alpha}} \frac{\tilde{d}^2}{R^2} \left(n + (\alpha - n) \frac{|X^T|^2}{R^2} \right).$$

By the first variation formula and Hölder's inequality it follows that

$$\alpha \int_{B_1} \zeta^2 \frac{\tilde{d}^2}{R^{n+2-\alpha}} \mathrm{d}\|V\|$$
$$\leq C \left(\int_{B_1} \frac{\zeta^2}{R^{n-\alpha}} \frac{\tilde{d}^2}{R^2} \mathrm{d}\|V\| \right)^{1/2} \left(\int_{B_1} \frac{\zeta^2}{R^{n+2-\alpha}} |X^\perp|^2 \mathrm{d}\|V\| \right)^{1/2}$$
$$+ C \left(\int_{B_1} \frac{\zeta^2}{R^{n-\alpha}} \frac{\tilde{d}^2}{R^2} \mathrm{d}\|V\| \right)^{1/2} \left(\int_{B_1} \frac{1}{R^{n-\alpha}} \tilde{d}^2 |\nabla^M \zeta|^2 \mathrm{d}\|V\| \right)^{1/2}$$
$$+ C \int_{B_1} \frac{\zeta^2}{R^{n+2-\alpha}} |X^\perp|^2 \mathrm{d}\|V\|.$$

Using the weighted Young's inequality we find

$$\int_{B_\gamma} \frac{d^2}{R^{n+2-\alpha}} \mathrm{d}\|V\| \leq C \int_{B_1} \frac{\zeta^2}{R^{n+2-\alpha}} |X^\perp|^2 \mathrm{d}\|V\| + C \int_{B_1} \frac{\tilde{d}^2}{R^{n-\alpha}} |\nabla^M \zeta|^2 \mathrm{d}\|V\|, \tag{3.3.14}$$

where $C = C(\mathbf{C}^{(0)}, \alpha)$. Applying (3.3.13) with γ replaced by $(1+\gamma)/2$ we have

$$\int_{B_1} \frac{\zeta^2}{R^{n+2-\alpha}} |X^\perp|^2 \mathrm{d}\|V\| \leq \int_{B_{(1+\gamma)/2}} \frac{|X^\perp|^2}{R^{n+2}} \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|.$$

Moreover, $|\nabla^M \zeta| \neq 0$ implies that $R \geq (1+\gamma)/2 \geq 1/2$, and $|\nabla^M \zeta| \leq C$, so it follows

$$\int_{B_1} \frac{d^2}{R^{n-\alpha}} |\nabla^M \zeta|^2 \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|.$$

Plugging these estimates back into (3.3.14) we obtain

$$\int_{B_\gamma} \frac{d^2}{R^{n+2-\alpha}} \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|, \tag{3.3.15}$$

which concludes the proof. □

The second ingredient in the proof of Theorem 3.3.1 is an analogue of [52, Lemma 3.9].

Lemma 3.3.5. *Let* $\mathbf{C}^{(0)} \in \mathcal{C}$. *There exists* $\varepsilon_0 = \varepsilon_0(\mathbf{C}^{(0)}) > 0$ *such that if* V, $\mathbf{C}$ *and* $\mathbf{C}^{(0)}$ *satisfy Hypotheses A with* $\varepsilon_A = \varepsilon_0$ *and* $\delta_A = 1/8$, *then for* $Z \in \mathrm{sing} V \cap B_{3/4}$ *with* $\Theta(\|V\|, Z) \geq \Theta(\|\mathbf{C}^{(0)}\|, 0) = 2$ *we have*

$$\mathrm{dist}^2(Z, B) + \int_{B_1} d_Z^2 \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|, \tag{3.3.16}$$

where $d(X) := \mathrm{dist}(X, \mathrm{spt}\|\mathbf{C}\|)$ *and* $d_Z(X) := \mathrm{dist}(X, \mathrm{spt}\|T_{Z\#}\mathbf{C}\|)$ *and* $C = C(\mathbf{C}^{(0)}, \gamma)$.

Proof. We assume throughout that $Z = (\xi, \eta) \in \mathrm{sing} V \cap B_{3/4}$, and that V, $\mathbf{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_0$ and $\delta_A = 1/8$, where $\varepsilon_0 > 0$ is to be chosen. Notice first of all that by the triangle inequality we have

$$|d(X) - d_Z(X)| \leq |\xi|, \tag{3.3.17}$$

and hence

$$\int_{B_1} d_Z^2 \mathrm{d}\|V\| \leq 2 \int_{B_1} d^2 \mathrm{d}\|V\| + C|\xi|^2, \tag{3.3.18}$$

where $C = C(\mathbf{C}^{(0)})$. Thus to prove (3.3.16), it suffices to show that

$$|\xi|^2 = \mathrm{dist}(Z, B) \leq C \int_{B_1} d^2 \mathrm{d}\|V\|.$$

By Lemma 3.2.5 we know that $|\xi| = \mathrm{dist}(X, \{0\} \times \mathbb{R}^{n-1}) \leq \tau$ with $\tau(\varepsilon_0, \mathbf{C}^{(0)}) \to 0$ as $\varepsilon_0 \searrow 0$. Therefore there exists $\theta = \theta(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$ such that if $X = (x, y) \in W := \mathrm{spt}\|V\| \cap (B_1(0) \setminus (B_\theta^{k+1}(0) \times \mathbb{R}^{n-1}))$, then

$$\mathrm{dist}(X, \mathrm{spt}\|T_{Z\#}\mathbf{C}\|) = |(x, y) - (x', y) - \xi^\perp|, \tag{3.3.19}$$

where $x' = p_{\mathbf{C}^{(0)}}(x)$ is the nearest point projection of x onto $\mathbf{C}^{(0)}$, and $\xi^\perp$ is the orthogonal projection of $(\xi, 0)$ onto $(T_{(x',0)}\mathbf{C})^\perp$. Indeed $X \in W$, if θ is chosen appropriately, implies that the nearest of the four half-planes making up $\mathbf{C}$ to X and $X + Z$ are the same. By the triangle inequality applied to (3.3.19) we have

$$|\xi^\perp| \leq d_Z(X) + d(X). \tag{3.3.20}$$

We now claim that there is $\delta_1 = \delta_1(\mathbf{C}^{(0)}) > 0$ and, given $\rho \in (0, 1/4)$, a constant $\varepsilon_0 = \varepsilon_0(\rho, \mathbf{C}^{(0)}) > 0$ such that if V, $\mathbf{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_0$

and $\delta_A = 1/4$, we have

$$\|V\| \left(\left\{ X \in B_\rho(Z) \cap W \,\middle|\, \delta_1 |a| \le |a^\perp| \right\} \right) \ge \delta_1 \rho^n, \tag{3.3.21}$$

for any $a \in \mathbb{R}^{n+k}$. Indeed, if this were not the case, then for every $\delta_1 > 0$, there would exist $\rho \in (0, 1/4)$ such that for each $i \ge 1$ there are corresponding V^i, $\mathbf{C}^i$ such that V^i, $\mathbf{C}^i$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = 1/i$ and $\delta_A = 1/4$, as well as sequences $Z_i \in \mathrm{sing} V^i$ and $a_i \in S^k$ such that

$$\|V^i\| \left(\left\{ X \in B_\rho(Z_i) \cap W_i \,\middle|\, \delta_1 \le |a_i^\perp| \right\} \right) < \delta_1 \rho^n,$$

where $W_i = \mathrm{spt}\|V^i\| \cap (B_1(0) \setminus (B^{k+1}_{\theta_i}(0) \times \mathbb{R}^{n-1}))$ with $\theta_i \to 0$. Passing to a subsequence we may assume that $V^i \to \mathbf{C}^{(0)}$, $Z_i \to Z \in B$ with $|Z| \le 3/4$ and $a_i \to a \in S^k$ so that

$$\|\mathbf{C}^{(0)}\| \left(\left\{ X \in B_\rho(Z) \,\middle|\, \delta_1 \le |a^\perp| \right\} \right) \le \delta_1 \rho^n. \tag{3.3.22}$$

Therefore if (3.3.21) were not true, then for all $\delta_1 > 0$ there exist $\rho > 0$, $Z \in B$ and $a \in S^k$ such that (3.3.22) holds. Translating and rescaling we can assume that $\rho = 1$, $Z = 0$, since $\mathbf{C}^{(0)}$ is translation invariant along B. Thus for each $j \ge 1$ we can choose $\delta_1 = 1/j$, then there exist corresponding $a_j \in S^k$ such that

$$\|\mathbf{C}^{(0)}\| \left(\left\{ X \in B_1(0) \,\middle|\, j^{-1} \le |a_j^\perp| \right\} \right) \le j^{-1}.$$

Passing again to a subsequence we can assume that $a_j \to a \in S^k$ with $a^\perp = 0$ $\|\mathbf{C}^{(0)}\|$-almost everywhere in B_1. This however is a contradiction as it implies that $\mathbf{C}^{(0)}$ is translation invariant in the a direction.

Thus we fix $\rho_0 \in (0, 1/8)$ and let $\varepsilon_0(\rho_0, \mathbf{C}^{(0)})$ be the corresponding ε_0 arising from the above discussion. By applying (3.3.21) we find

$$\rho_0^n |\xi|^2 \le C \int_{W \cap B_{\rho_0}(Z)} |\xi^\perp|^2 \mathrm{d}\|V\|.$$

Combining this with (3.3.20) we therefore have

$$\rho_0^n |\xi|^2 \le C \int_{B_{\rho_0}(Z)} d_Z^2 \mathrm{d}\|V\| + C \int_{B_1} d^2 \mathrm{d}\|V\|.$$

We now apply Lemma 3.3.4 with $\alpha = 1/2$ to $\eta_{Z,1/4\#}V$ and $\mathbf{C}$. Provided ε_0 is chosen sufficiently small, Lemma 3.2.4 and (3.3.17) combined with the fact that $|\xi| \leq \tau(\varepsilon_0)$ imply that the assumptions of Lemma 3.3.4 are satisfied. We therefore have that

$$\frac{1}{\rho_0^{n+3/2}} \int_{B_{\rho_0}(Z)} d_Z^2 \mathrm{d}\|V\| \leq C \int_{B_{1/4}(Z)} d_Z^2 \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\| + C|\xi|^2.$$

Note in particular that $C = C(\mathbf{C}^{(0)})$ is independent of ρ_0. Thus

$$|\xi|^2 \rho_0^n \leq C \int_{B_1} d^2 \mathrm{d}\|V\| + C\rho_0^{n+3/2} |\xi|^2.$$

So by choosing ρ_0 suitably small (e.g. such that $C\rho_0^{3/2} \leq 1/2$), we find

$$|\xi|^2 \leq C \int_{B_1} d^2 \mathrm{d}\|V\|,$$

provided $\varepsilon_0 = \varepsilon_0(\mathbf{C}^{(0)})$ is sufficiently small. □

Combining Lemmas 3.3.4 and 3.3.5 we may now prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Let $Z = (\xi, \eta) \in \mathrm{spt}\|V\| \cap B_{3/4}$ with $\Theta(\|V\|, Z) \geq 2$. By Lemma 3.3.5 we have

$$|\xi|^2 \leq C \int_{B_1} d^2 \mathrm{d}\|V\|. \tag{3.3.23}$$

Therefore, if ε_0 is small enough, we may apply Lemma 3.3.4 to $\eta_{Z,1/4\#}V$ with $\alpha = 1/4$, $\gamma = 1/2$ and deduce from the triangle inequality that

$$\int_{B_{1/8}(Z)} \frac{d_Z^2}{|X - Z|^{n+7/4}} \mathrm{d}\|V\| \leq C \int_{B_{1/4}(Z)} d_Z^2 \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|. \tag{3.3.24}$$

This of course implies

$$\int_{B_{1/8}(Z)} \frac{d_Z^2}{|X - Z|^{n-1/4}} \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|. \tag{3.3.25}$$

Because $d \leq d_Z + |\xi|$, we have

$$\int_{B_{1/8}(Z)} \frac{d^2}{|X - Z|^{n-1/4}} \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|.$$

Since we have the estimate $|X - Z|^{-(n-1/4)} \leq 8^{n-1/4}$ on $\mathrm{spt}\|V\| \setminus B_{1/8}(Z)$, it readily follows that

$$\int_{B_\gamma} \frac{d^2}{|X - Z|^{n-1/4}} \mathrm{d}\|V\| \leq C \int_{B_1} d^2 \mathrm{d}\|V\|.$$

The estimates on the remaining terms appearing on the left hand side of inequality (3.3.1a) follow directly from Lemmas 3.3.4 and 3.3.5.

To establish (3.3.1b) we need only observe that on the set $U_\tau := \{X = (x, y) \in \mathrm{spt}\|\mathbf{C}\| \cap B_\gamma \mid |x| > \tau\}$, we have $d_Z(X + u(X)) = |u(X) - \xi^\perp(X)|$, which follows from Lemma 3.2.6. Hence applying (3.3.24) we have

$$\int_{U_\tau \cap B_{1/8}(Z)} \frac{|u(X) - \xi^\perp(X)|^2}{|X - Z|^{n+7/4}} \mathrm{d}\mathcal{H}^n \leq C \int_{B_1} d_Z^2 \mathrm{d}\|V\|.$$

As before we can bound $|X - Z|^{-n-7/4}$ on the set $U_\tau \setminus B_{1/8}(Z)$ to establish (3.3.1b). □

3.4 Blow-ups

In this section we construct blow-ups, which represent solutions of the linearised problem. The first step is to show an abundance of points with density at least 2, which is required to apply the estimates of Corollary 3.3.2 and deduce that excess does not concentrate near the axis. Recall Definition 3.1.6 from Section 3.1.1. We say $V \in \mathcal{V}$ if V has no triple junction singularities in $B_1(0) \setminus B$, and the orthogonal projection of $\mathrm{sing} V \cap B_1$ to B has full $\mathcal{H}^{n-1}$-measure. We can show that members of the class $\mathcal{V}$ necessarily have an abundance of singularities with density at least 2 near the axis. We refer to such points as 'good density points'.

Lemma 3.4.1. *Let $\delta > 0$ and $\mathbf{C}^{(0)} \in \mathcal{C}$. There is $\varepsilon = \varepsilon(\mathbf{C}^{(0)}, \delta) > 0$ such that if $V \in \mathcal{V}$, $\mathbf{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon$ and $\delta_A = 1/4$ then*

$$B \cap \{(x, y) \mid |y| \leq 1\} \subset B_\delta \left(\{X \mid \Theta(\|V\|, X) \geq 2\}\right). \tag{3.4.1}$$

Proof. By Lemma 3.2.5 we may choose ε small enough depending on $\mathbf{C}^{(0)}$ and δ such that $(\mathrm{spt}\|V\| \cap B_{3/2}) \setminus U_\delta$ decomposes into smooth single-valued graphs over $\mathbf{C}$. Hence, by (M2), it follows that for $\mathcal{H}^{n-1}$-almost every $y \in B_1^{n-1}$ there exists $(x,y) \in \text{sing}V \cap B_1$ such that $|x| < \delta$. By stratification of the singular set (Lemma 2.3.6), $\mathcal{H}^{n-1}$-almost every such (x,y) has a tangent cone that is either a multiplicity 2 or higher plane, or a tangent cone with a one-dimensional cross section. In the former case we trivially have $\Theta(\|V\|, X) \geq 2$. In the latter case, the one-dimensional cross section of this cone must consist of three or more half-lines meeting at a common point. We need only rule out the possibility that it consists of three half lines. In this case Simon [52] tells us that in a small ball centred on X, the varifold consists of three smooth sheets coming together along a common boundary, i.e. a triple junction. If $x \neq 0$ then this is an immediate contradiction of (M1). Otherwise, since $\text{spt}\|V\|$ consists of three smooth sheets in a neighbourhood of x, and four smooth sheets away from the axis, there must exist another singularity $(\tilde{x}, y)$ with $\tilde{x} \neq 0$ to which we may apply the above reasoning. □

3.4.1 Constructing blow-ups

Let $\mathbf{C}^{(0)} \in \mathcal{C}$, $\varepsilon_j \searrow 0$ and suppose that $V^j \in \mathcal{V}$, $\mathbf{C}^j \in \mathcal{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_j$ and $\delta_A = 1/16$. Thus we assert the following.

(1_j) Each V^j satisfies the mass bound

$$\frac{\|V^j\|(B_{R_0}(0))}{R_0^n \omega_n} \leq 2 + \frac{1}{16}.$$

(2_j) The height excess $E_{V^j}(\mathbf{C}^j)$ of V^j relative to $\mathbf{C}^j$, which we denote by E_j, satisfies

$$E_j^2 := E_{V^j}^2(\mathbf{C}^j) = \int_{B_1} \text{dist}^2(X, \text{spt}\|\mathbf{C}^j\|) \text{d}\|V^j\| \leq Q_{V^j}^2(\mathbf{C}^j) < \varepsilon_j^2.$$

(3_j) $\mathbf{C}^j$ is Hausdorff close to $\mathbf{C}^{(0)}$, that is

$$\text{dist}_{\mathcal{H}}(\text{spt}\|\mathbf{C}^j\| \cap B_1,\ \text{spt}\|\mathbf{C}^{(0)}\| \cap B_1) < \varepsilon_j.$$

(4_j) V^j has no triple junction singularities in $B_1(0) \setminus B$, and the orthogonal projection of $\text{sing}V^j \cap B_1$ to B has full $\mathcal{H}^{n-1}$-measure.

We further assume that $\Theta(\|V^j\|, 0) \geq 2$ for each j. Notice that if this weren't the case, then Lemma 3.4.1 implies the existence of $Z_j \in \operatorname{spt}\|V^j\|$ such that $\Theta(\|V^j\|, Z_j) \geq 2$ and $Z_j \to 0$. By translating Z_j to the origin, and possibly rescaling slightly to ensure that the translation remains in $\mathcal{V}$, we can ensure the existence of a good density point at the origin. Furthermore, since V^j, $\mathbf{C}^j$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_j$ and $\delta_A = 1/16$, we can ensure the modified sequences still satisfies Hypotheses A with $\varepsilon_A = \tilde{\varepsilon}_j$, and $\delta_A = 1/8$, where $\tilde{\varepsilon}_j \searrow 0$. This means we can still apply the results of Section 3.3 in the remainder of the construction.

We now pick sequences δ_j and τ_j going to 0 sufficiently slowly (depending on ε_j) that we may apply Lemma 3.4.1 with δ_j replacing δ in the statement, and Lemma 3.2.6 and Theorem 3.3.1 with τ_j replacing τ in the respective statements. Thus it follows, for sufficiently large j, that

(A_j) From Lemma 3.4.1 we know that for every $Y = (0, y) \in B \cap B_1$ we have

$$B_{\delta_j}(0, y) \cap \{Z \mid \Theta(\|V^j\|, Z) \geq 2\} \neq \emptyset.$$

Consequently Corollary 3.3.2 says that for every $\sigma \in [\delta_j, 1/4)$ we have

$$\int_{(B_\sigma^{k+1} \times \mathbb{R}^{n-1}) \cap B_{1/2}} \operatorname{dist}^2(X, \operatorname{spt}\|\mathbf{C}^j\|) \mathrm{d}\|V^j\| \leq C\sigma^{1/2} E_j^2. \tag{3.4.2}$$

(B_j) Lemma 3.2.5 implies that V^j admits a graphical decomposition of the form

$$V^j \llcorner (B_{3/2}(0) \cap \{r > \tau_j\}) = |\operatorname{graph}(u_j)| \llcorner (B_{3/2}(0) \cap \{r > \tau_j\}),$$

where $u_j \in C^\infty(\operatorname{spt}\|\mathbf{C}^j\| \cap \{r > \tau_j/2\} \cap B_{3/2}; (\operatorname{spt}\|\mathbf{C}^j\|)^\perp)$ is a smooth solution of the minimal surface system.

(C_j) By Lemma 3.3.5, for each point $Z_j = (\xi_j, \eta_j) \in \operatorname{spt}\|V^j\| \cap B_{3/4}$ with $\Theta(\|V^j\|, Z_j) \geq 2$ we have

$$|\xi_j|^2 = \operatorname{dist}^2(Z, B) \leq CE_j^2. \tag{3.4.3}$$

(D_j) For each $\rho \in (0, 1/4]$ there is a $J(\rho)$ such that for each $j \geq J$, and any

$Z \in \operatorname{spt}\|V^j\| \cap B_{3/8}$ we have, by Lemma 3.3.4 applied to $\eta_{Z,\rho\#}V$, implies

$$\int_{U_{\tau_j}\cap B_{\rho/2}(Z)} \frac{|u_j(X) - \xi^\perp(X)|^2}{|X-Z|^{n+7/4}} \mathrm{d}\mathcal{H}^n \leq \frac{C}{\rho^{n+7/4}} \int_{B_\rho(Z)} d_Z^2 \mathrm{d}\|V^j\|. \tag{3.4.4}$$

This is established by arguing as in the proof of Theorem 3.3.1.

We now define the functions

$$v_j := E_j^{-1} u_j.$$

For each compact $K \subset B_1 \setminus B$, it follows from Lemma 3.2.6 and standard elliptic estimates that

$$\sup_{K\cap \mathbf{C}^j} |\nabla^i v_j| \leq C, \qquad i = 0,1,2,3, \tag{3.4.5}$$

for j sufficiently large depending on K. Moreover since we have that $\mathbf{C}^j \in \mathcal{C}$ and $\operatorname{dist}_{\mathcal{H}}(\operatorname{spt}\|\mathbf{C}^j\| \cap B_1,\, \operatorname{spt}\|\mathbf{C}^{(0)}\| \cap B_1) < \varepsilon_j$ we know

$$\operatorname{spt}\|\mathbf{C}^j\| \cap B_{3/2} \subset \operatorname{graph}(\psi_j)$$

for some $\psi_j \in C^2(\operatorname{spt}\|\mathbf{C}^{(0)}\| \cap B_{3/2}; (\operatorname{spt}\|\mathbf{C}^{(0)}\|)^\perp)$ that is linear on each $H_i^{(0)}$ and with $\|\psi_j\|_{C^2} \leq C\varepsilon_j \to 0$. This combined with (3.4.5) implies that, up to extracting a subsequence, $v_j(x + \psi_j(x))$ converges in C^2_{loc} on $\operatorname{spt}\|\mathbf{C}^{(0)}\| \cap B_1$ to a limit function $v \in C^2(\operatorname{spt}\|\mathbf{C}^{(0)}\| \cap \{r > 0\} \cap B_1; (\operatorname{spt}\|\mathbf{C}^{(0)}\|)^\perp)$. Moreover, (3.4.2) implies that for $\sigma \in (0, 1/4)$, we have

$$\int_{\operatorname{spt}\|\mathbf{C}^{(0)}\|\cap\{0<r<\sigma\}\cap B_{1/2}} |v|^2 \mathrm{d}\mathcal{H}^n \leq C\sigma^{1/2},$$

from which it follows that convergence is also strong in L^2 for every $\sigma \in (0,1)$ on the set $\operatorname{spt}\|\mathbf{C}^{(0)}\| \cap \{r > 0\} \cap B_\sigma(0)$. We let $\Omega := \operatorname{spt}\|\mathbf{C}^{(0)}\| \cap \{r > 0\} \cap B_1(0)$ for brevity, and we make the following definition.

Definition 3.4.2 (Blow-up)**.** *Corresponding to* $\mathbf{C}^{(0)}$*, and sequences* $\{\mathbf{C}^j\} \subset \mathcal{C}$*,* $\{V^j\} \subset \mathcal{V}$*, and* $\{\varepsilon_j\}$ *such that* V^j*,* $\mathbf{C}^j$ *and* $\mathbf{C}^{(0)}$ *satisfy Hypotheses A with* $\varepsilon_A = \varepsilon_j$ *and* $\delta_A = 1/16$*, we call any* $v \in L^2(\Omega; (\operatorname{spt}\|\mathbf{C}^{(0)}\|)^\perp) \cap C^\infty(\Omega; (\operatorname{spt}\|\mathbf{C}^{(0)}\|)^\perp)$ *constructed in this way a* blow-up of the sequence V^j off $\mathbf{C}^{(0)}$ relative to $\mathbf{C}^j$*. We denote the class of all blow-ups off* $\mathbf{C}^{(0)}$ *by* $\mathfrak{B}(\mathbf{C}^{(0)})$*.*

3.4.2 Properties of blow-ups

In this section we analyse the class $\mathfrak{B}(\mathbf{C}^{(0)})$ and prove basic fundamental properties of the members which will in turn imply strong regularity properties.

Lemma 3.4.3 (Properties of blow-ups)**.** *Given $\mathbf{C}^{(0)} \in \mathcal{C}$, any $v \in \mathfrak{B}(\mathbf{C}^{(0)})$ satisfies the following properties.*

($\mathfrak{B}1$) $v \in L^2(\Omega; (\mathrm{spt}\|\mathbf{C}^{(0)}\|)^\perp) \cap C^\infty(\Omega; (\mathrm{spt}\|\mathbf{C}^{(0)}\|)^\perp)$.

($\mathfrak{B}2$) $\Delta v = 0$ on Ω.

($\mathfrak{B}3$) For each $Y \in B \cap B_{5/16}(0)$, there is $\kappa(Y) \in B$ satisfying $\kappa(0) = 0$, $|\kappa(Y)|^2 \leq C \int_{\Omega \cap B_{1/2}} |v|^2 \mathrm{d}\mathcal{H}^n$, for some constant $C = C(\mathbf{C}^{(0)})$, and moreover for each $\rho \in (0, 1/8]$ we have the estimate

$$\int_{B_{\rho/2}(Y)\cap\Omega} \frac{|v(X) - \kappa^\perp(Y)|^2}{|X - Y|^{n+3/2}} \mathrm{d}\mathcal{H}^n \leq \frac{C}{\rho^{n+3/2}} \int_{B_\rho(Y)\cap\Omega} |v(X) - \kappa^\perp(Y)|^2 \mathrm{d}\mathcal{H}^n. \tag{3.4.6}$$

Proof. Properties ($\mathfrak{B}1$) and ($\mathfrak{B}2$) follow directly from the construction.

To see ($\mathfrak{B}3$), we first let $v \in \mathfrak{B}(\mathbf{C}^{(0)})$ and suppose that v is a blow-up of V^j off $\mathbf{C}^{(0)}$ relative to $\mathbf{C}^j$. Fix $Y \in B \cap B_{5/16}(0)$ and choose a sequence $Z_j \in \mathrm{spt}\|V^j\| \cap B_{3/8}(0)$ with $\Theta(\|V^j\|, Z_j) \geq 2$ and $Z_j \to Y$. The existence of such a sequence is evidently guaranteed by (A_j). Then from (3.4.4) it follows that

$$\int_{U_{\tau_j} \cap B_{\rho/2}(Z_j)} \frac{|u_j(X) - \xi_j^\perp(X)|^2}{|X - Z|^{n+3/2}} \mathrm{d}\mathcal{H}^n \leq \frac{C}{\rho^{n+3/2}} \int_{B_\rho(Z_j)} d_{Z_j}^2 \mathrm{d}\|V^j\|. \tag{3.4.7}$$

Notice that (C_j) implies, that after possibly passing to a subsequence which we do not relabel, there is $\kappa(Y) \in B$ such that

$$\lim_{j\to\infty} E_j^{-1}(\xi_j, 0) \to \kappa(Y).$$

In fact, by applying Lemma 3.3.5 to $\eta_{0,1/2\#}V^j$ we conclude that

$$|\xi_j|^2 \leq C \int_{B_{1/2}} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}^j\|) \mathrm{d}\|V^j\|.$$

On dividing both sides by E_j^2 and letting $j \to \infty$ we find

$$|\kappa(Y)|^2 \leq \int_{\Omega \cap B_{1/2}} |v|^2 \mathrm{d}\mathcal{H}^n.$$

Moreover, dividing both sides of (3.4.7) by E_j^2, letting $j \to \infty$ and invoking (3.4.2) we establish (3.4.6). Notice that (3.4.6) implies that $\kappa(Y)$ depends only on Y and not on the sequence Z_j used to construct it, justifying our notation. Finally observe that if $Y = 0$, we can choose the sequence $Z_j = 0$ from which it evidently follows that $\kappa(0) = 0$. □

3.4.3 Regularity of blow-ups

In this section we prove that blow-ups are globally $C^{1,1}$ up to the axis, and hence boundary Schauder theory will give us excess decay for blow-ups. We first show that blow-ups are $C^{0,\alpha}(\overline{\Omega})$. The argument is based on Campanato style estimates, used in a similar fashion by Wickramasekera in [64]

Lemma 3.4.4. *Let $v \in \mathfrak{B}(\mathbf{C}^{(0)})$. Then $v \in C^{0,\alpha}(\overline{B_{5/16} \cap \{r > 0\}}; (\mathrm{spt}\|\mathbf{C}^{(0)}\|)^{\perp})$ for some $\alpha = \alpha(\mathbf{C}^{(0)}) \in (0,1)$ and the following estimate holds*

$$\sup_{\overline{B_{5/16} \cap \Omega}} |v|^2 + \sup_{\substack{X,\, Y \in \overline{B_{5/16} \cap \Omega} \\ X \neq Y}} \frac{|v(X) - v(Y)|^2}{|X - Y|^{2\alpha}} \leq C \left(\int_{\Omega \cap B_{1/2}} |v|^2 \mathrm{d}\mathcal{H}^n \right). \tag{3.4.8}$$

Proof. Let $v \in \mathfrak{B}(\mathbf{C}^{(0)})$ and let $Y \in B \cap B_{5/16}$. Then property ($\mathfrak{B}3$) of Lemma 3.4.3 and (3.4.6) tell us that for any $\rho \in (0, 1/8]$

$$\int_{\Omega \cap B_{\rho/2}(Y)} \frac{|v(X) - \kappa(Y)|^2}{|X - Y|^{n+3/2}} \mathrm{d}\mathcal{H}^n \leq \frac{C}{\rho^{n+3/2}} \int_{B_\rho(Y) \cap \Omega} |v(X) - \kappa(Y)|^2 \mathrm{d}\mathcal{H}^n, \tag{3.4.9}$$

and that

$$|\kappa(Y)|^2 \leq C \int_{B_{1/2} \cap \{r > 0\}} |v|^2 \mathrm{d}\mathcal{H}^n,$$

where $C = C(\mathbf{C}^{(0)})$. As observed previously, it follows immediately from (3.4.9) that $\kappa(Y)$ is unique and depends only on the point Y and not on the sequence of singular points Z_j used to construct it. Therefore we may declare $v(Y) := \kappa(Y)$. With this choice of boundary values along the axis, we seek to prove that v is $C^{0,\alpha}$. Now, the estimate (3.4.9) implies that for a fixed constant $C = C(\mathbf{C}^{(0)})$, and for $0 < \sigma \leq \rho/2 \leq 1/32$ we have

$$\frac{1}{\sigma^n}\int_{B_\sigma(Y)\cap\Omega}|v(X)-v(Y)|^2 \mathrm{d}\mathcal{H}^n \leq C\left(\frac{\sigma}{\rho}\right)^{3/2}\frac{1}{\rho^n}\int_{B_\rho(Y)\cap\Omega}|v(X)-v(Y)|^2 \mathrm{d}\mathcal{H}^n. \tag{3.4.10}$$

Let $Z \in B_{5/16}\cap\{r>0\}$, $\rho \in (0,16]$ and let $Y \in B$ be the unique point such that $|Z-Y| = \mathrm{dist}(Z,B)$. We take $\gamma \in (0,1/16]$ to be determined, and suppose that $\mathrm{dist}(Z,B) < \gamma\rho$, then

$$\begin{aligned}
&\frac{1}{(\gamma\rho)^n}\int_{B_{\gamma\rho}(Z)\cap\Omega}|v-v(Y)|^2 \mathrm{d}\mathcal{H}^n \\
&\quad\leq \left(\frac{2}{\gamma\rho+|Z-Y|}\right)^n\int_{B_{\gamma\rho+|Z-Y|}(Y)\cap\Omega}|v-v(Y)|^2 \mathrm{d}\mathcal{H}^n \\
&\quad\leq 2^nC\left(\frac{\gamma\rho+|Z-Y|}{\rho-|Z-Y|}\right)^{3/2}\frac{1}{(\rho-|Z-Y|)^n}\int_{B_{\rho-|Z-Y|}(Y)\cap\Omega}|v-v(Y)|^2 \mathrm{d}\mathcal{H}^n \\
&\quad\leq 4^nC\left(\frac{2\gamma}{1-\gamma}\right)^{3/2}\frac{1}{\rho^n}\int_{B_\rho(Z)\cap\Omega}|v-v(Y)|^2 \mathrm{d}\mathcal{H}^n,
\end{aligned}$$

where we applied (3.4.10) with σ replaced by $\gamma\rho+|Z-Y|$ and ρ replaced by $\rho-|Z-Y|$. We now choose $\gamma = \gamma(\mathbf{C}^{(0)}) \in (0,1/16]$ to be such that

$$4^nC\left(\frac{2\gamma}{1-\gamma}\right)^{3/2} < 1/4,$$

then

$$\frac{1}{(\gamma\rho)^n}\int_{B_{\gamma\rho}(Z)\cap\Omega}|v-v(Y)|^2 \mathrm{d}\mathcal{H}^n \leq \frac{1}{4\rho^n}\int_{B_\rho(Z)\cap\Omega}|v-v(Y)|^2 \mathrm{d}\mathcal{H}^n \tag{3.4.11}$$

for any $Z \in B_{5/16}$ such that $|Z-Y| = \mathrm{dist}(Z,B) < \gamma\rho$. Conversely, if $\gamma\rho \leq \mathrm{dist}(Z,B)$, then since v is harmonic in Ω, standard elliptic estimates imply that for all $\sigma \in (0,1/2]$ and $b \in (\mathrm{spt}\|\mathbf{C}^{(0)}\|)^\perp$ the following estimate holds

$$\frac{1}{(\sigma\gamma\rho)^n}\int_{B_{\sigma\gamma\rho}(Z)\cap\mathbf{C}^{(0)}}|v-v(Z)|^2 \mathrm{d}\mathcal{H}^n \leq \frac{C\sigma^2}{(\gamma\rho)^n}\int_{B_{\gamma\rho}(Z)\cap\mathbf{C}^{(0)}}|v-b|^2 \mathrm{d}\mathcal{H}^n. \tag{3.4.12}$$

Now fix $Z \in B_{5/16}\cap\Omega$ and let j^* be such that

$$\gamma^{j^*+1} < \mathrm{dist}(Z,B) \leq \gamma^{j^*}.$$

Then with $Y \in B$ such that $|Z - Y| = \operatorname{dist}(Z, B)$ we have from (3.4.12) that

$$\frac{1}{(\sigma\gamma^{j^*+1})^n} \int_{B_{\sigma\gamma^{j^*+1}}(Z)\cap\Omega} |v - v(Z)|^2 \mathrm{d}\mathcal{H}^n \leq \frac{C\sigma^2}{(\gamma^{j^*+1})^n} \int_{B_{\gamma^{j^*+1}}(Z)\cap\Omega} |v - v(Y)|^2 \mathrm{d}\mathcal{H}^n. \tag{3.4.13}$$

Furthermore, if $j^* \geq 1$ then by iterating (3.4.11) we have

$$\begin{aligned} \frac{1}{(\gamma^j)^n} \int_{B_{\gamma^j}(Z)\cap\Omega} |v - v(Y)|^2 \mathrm{d}\mathcal{H}^n &\leq \frac{1}{4(\gamma^{j-1})^n} \int_{B_{\gamma^{j-1}}(Z)\cap\Omega} |v - v(Y)|^2 \mathrm{d}\mathcal{H}^n \\ &\leq \frac{1}{4^{j-1}\gamma^n} \int_{B_\gamma(Z)\cap\Omega} |v - v(Y)|^2 \mathrm{d}\mathcal{H}^n \end{aligned} \tag{3.4.14}$$

for each $j = 1, \ldots, j^*$. On the other hand, letting $j = j^*$ in (3.4.14) and choosing $\sigma = 1/2$ in (3.4.13) we get

$$\begin{aligned} |v(Z) - v(Y)|^2 = \omega_n^{-1} \left(\frac{1}{2}\gamma^{j^*+1}\right)^{-n} \Bigg(&\int_{B_{\gamma^{j^*+1}/2}(Z)\cap\Omega} |v - v(Z)|^2 \mathrm{d}\mathcal{H}^n \\ &+ \int_{B_{\gamma^{j^*+1}/2}(Z)\cap\Omega} |v - v(Y)|^2 \mathrm{d}\mathcal{H}^n \Bigg) \\ \leq \frac{C}{(\gamma^{j^*+1})^n} \int_{B_{\gamma^{j^*+1}}(Z)\cap\Omega} |v - v(Y)|^2 & \\ &+ \frac{C}{(\gamma^{j^*+1})^n} \int_{B_{\gamma^{j^*}}(Z)\cap\Omega} |v - v(Y)|^2 \mathrm{d}\mathcal{H}^n \\ \leq \frac{C}{(\gamma^{j^*})^n} \int_{B_{\gamma^{j^*}}(Z)\cap\Omega} |v - v(Y)|^2 \mathrm{d}\mathcal{H}^n & \\ \leq \frac{C}{4^{j^*-1}} \int_{B_\gamma(Z)\cap\Omega} |v - v(Y)|^2 \mathrm{d}\mathcal{H}^n, & \end{aligned}$$

where $C = C(\mathbf{C}^{(0)}, \gamma)$. Therefore using (3.4.14) and the triangle inequality we recover

$$\frac{1}{(\gamma^j)^n} \int_{B_{\gamma^j}(Z)\cap\Omega} |v - v(Z)|^2 \mathrm{d}\mathcal{H}^n \leq \frac{C}{4^{j-1}} \int_{B_\gamma(Z)\cap\Omega} |v - v(Y)|^2 \mathrm{d}\mathcal{H}^n \tag{3.4.15}$$

for $j = 1, \ldots, j^*$ and where $C = C(\mathbf{C}^{(0)}, \gamma)$. Now let $\rho \in (0, \gamma/2]$. Then if $2\rho \leq \gamma^{j^*+1}$ we may write $\rho = \sigma\gamma^{j^*+1}$ for some $\sigma \in (0, 1/2]$, and so by (3.4.13) and the bound on $|\kappa(Y)|^2$ we have

$$\frac{1}{\rho^n}\int_{B_\rho(Z)\cap\Omega}|v-v(Z)|^2\mathrm{d}\mathcal{H}^n \leq \frac{C\sigma^2}{4^{j^*}}\int_{B_\gamma(Z)\cap\Omega}|v-v(Y)|^2\mathrm{d}\mathcal{H}^n \leq \frac{C\sigma^2}{4^{j^*+1}}\int_{B_{1/2}(Z)\cap\Omega}|v|^2\mathrm{d}\mathcal{H}^n.$$

Alternatively, there is a $j \in \{1,\dots,j^*\}$ such that $\gamma^{j+1} < 2\rho \leq \gamma^j$. Then from (3.4.15)

$$\frac{1}{\rho^n}\int_{B_\rho(Z)\cap\Omega}|v-v(Z)|^2\mathrm{d}\mathcal{H}^n \leq \frac{C}{4^{j-1}}\int_{B_\gamma(Z)\cap\Omega}|v-v(Y)|^2\mathrm{d}\mathcal{H}^n \leq \frac{C}{4^{j+1}}\int_{B_{1/2}(Z)\cap\Omega}|v|^2\mathrm{d}\mathcal{H}^n.$$

Observing that $(\sigma 2^{-(j^*+1)})^2 \leq \rho^{2\lambda}$ in the former case, and $4^{j+1} \leq \rho^{2\alpha}$ in the latter case for some appropriately chosen $\alpha = \alpha(\mathbf{C}^{(0)})$ we have, for any given $Z \in B_{5/16} \cap \operatorname{spt}\|\mathbf{C}^{(0)}\|$ and $\rho \in (0,\gamma/2]$, the estimate

$$\frac{1}{\rho^n}\int_{B_\rho(Z)\cap\Omega}|v-v(Z)|^2\mathrm{d}\mathcal{H}^n \leq C\rho^{2\alpha}\int_{B_{1/2}\cap\Omega}|v|^2\mathrm{d}\mathcal{H}^n. \tag{3.4.16}$$

From here, Hölder continuity of v in $\overline{B_{5/16}\cap\Omega}$ follows easily. Indeed given Z_1, $Z_2 \in \overline{B_{5/16}\cap\Omega}$ let $\rho := |Z_1 - Z_2|$. If $2\rho \leq \gamma/2$, then noting that both $B_\rho(Z_1)$, $B_\rho(Z_2) \subset B_{2\rho}(Z_1)\cap B_{2\rho}(Z_2)$ we obtain

$$\begin{aligned}|v(Z_1)-v(Z_2)|^2 &\leq \frac{C}{\rho^n}\int_{B_{2\rho}(Z_1)\cap B_{2\rho}(Z_2)\cap\Omega}|v-v(Z_1)|^2+|v-v(Z_2)|^2\mathrm{d}\mathcal{H}^n\\ &\leq \frac{C}{\rho^n}\int_{B_{2\rho}(Z_1)\cap\Omega}|v-v(Z_1)|^2\mathrm{d}\mathcal{H}^n + \frac{C}{\rho^n}\int_{B_{2\rho}(Z_2)\cap\Omega}|v-v(Z_2)|^2\mathrm{d}\mathcal{H}^n\\ &\leq C\rho^{2\alpha}\int_{B_{1/2}\cap\Omega}|v|^2\mathrm{d}\mathcal{H}^n.\end{aligned}$$

Hence

$$\frac{|v(Z_1)-v(Z_2)|^2}{|Z_1-Z_2|^{2\alpha}} \leq C\int_{B_{1/2}\cap\Omega}|v|^2\mathrm{d}\mathcal{H}^n.$$

Bearing in mind that $v(0) = 0$, the conclusions of the lemma now follow easily from here. □

Next we show that we can improve the regularity to C^2 with $C^{1,1}$-estimates controlled by excess.

Theorem 3.4.5. *Let* $v \in \mathfrak{B}(\mathbf{C}^{(0)})$*. Then* $v \in C^2(\overline{\Omega \cap B_{1/4}}; (\mathrm{spt}\|\mathbf{C}^{(0)}\|)^{\perp})$ *and we have the estimate*

$$\sup_{\overline{B_{1/4}\cap\Omega}} |Dv|^2 + \sup_{\substack{X,\,Y \in \overline{B_{1/4}\cap\Omega} \\ X \neq Y}} \frac{|Dv(X) - Dv(Y)|^2}{|X-Y|^2} \leq C \int_{B_{1/2}\cap\Omega} |v|^2 \mathrm{d}\mathcal{H}^n. \tag{3.4.17}$$

Proof. Fix some $v \in \mathfrak{B}(\mathbf{C}^{(0)})$. By definition there is a sequence $\varepsilon_j \searrow 0$ such that v is the blow-up of some sequence V^j off $\mathbf{C}^{(0)}$ with respect to $\mathbf{C}^j$. Choose $\zeta = \zeta(r,y)$ smooth such that $\zeta(r,y) \equiv 0$ if $r^2 + |y|^2 \geq 3/8$. In addition, we also suppose that $\partial\zeta/\partial r \equiv 0$ in a neighbourhood of $\{r = 0\}$, which is to say we assume the existence of some $\tau > 0$ such that

$$D_q\zeta(|x|, y) = 0, \qquad \text{for } |x| < 2\tau,\ \ q = 1, \ldots, k+1. \tag{3.4.18}$$

For each $a = 1, \ldots, n-1$ and $i = 1, \ldots, k+1$ the first variation formula with $\Phi := e_i\zeta_a$, where $e_i\zeta_a := e_i\partial\zeta/\partial y^a$, applied along the sequence V^j gives

$$\int_{B_1} \nabla^{M^j} x^i \cdot \nabla^{M^j} \zeta_a \mathrm{d}\|V^j\| = \int_{B_1} e_i \cdot \nabla^{M^j}\zeta_a \mathrm{d}\|V^j\| = 0.$$

We let $U_j := B_1 \setminus (B^{k+1}_{\tau_j} \times \mathbb{R}^{n-1}) \cap \mathrm{spt}\|\mathbf{C}^j\|$ where $\tau_j \searrow 0$ sufficiently slowly that $\mathrm{spt}\|V^j\| \setminus (B^{k+1}_{\tau_j} \times \mathbb{R}^{n+k})$ is single-valued graph of some C^2 function u_j over $\mathbf{C}^{(0)}$; note that such a sequence τ_j is guaranteed to exist by Lemma 3.2.6. We further define $G_j := \mathrm{graph}(u_j|_{U_j \cap B_{1/2}})$. Finally, we define $(g_j^{ip}(X))$ to be the matrix of the orthogonal projection onto $T_X M^j$, which exists at almost every point. We have

$$\begin{aligned} \int_{B_{1/2}\setminus G_j} & |\nabla^{M^j} x_i \cdot \nabla^{M^j}\zeta_a| \mathrm{d}\|V^j\| \\ &= \int_{B_{1/2}\setminus G_j} \left| \sum_{p=1}^{n+k} g_j^{ip}(X) D_p\zeta_a \right| \mathrm{d}\|V^j\| \\ &= \int_{B_{1/2}\setminus G_j} \left| \sum_{p=1}^{n-1} (\delta_{i,1+k+p} - g^{i,1+k+p}(X)) D_{y^p}\zeta_a \right| \mathrm{d}\|V^j\|, \end{aligned} \tag{3.4.19}$$

where we used the fact that $D_j\zeta_a = 0$ for each $j = 1, \ldots, k+1$ and that for

$i = 1, \ldots, k+1$ we have $\delta_{i,1+k+p} = 0$ for each p. Thus

$$\begin{aligned}
&\int_{B_{1/2}\setminus G_j} |\nabla^{M^j} x_i \cdot \nabla^{M^j} \zeta_a| \mathrm{d}\|V^j\| \\
&\quad \leq \int_{B_{1/2}\setminus G_j} \left(\sum_{p=1}^{n-1} |e_{1+k+p}^{\perp}|^2 \right)^{1/2} |D\zeta_a| \mathrm{d}\|V^j\| \\
&\quad \leq \sup_{B_{1/2}} |D\zeta_a| \left(\|V^j\|(B_{1/2} \setminus G_j) \right)^{1/2} \left(\int_{B_{1/2}\setminus G_j} \sum_{p=1}^{n-1} |e_{1+k+p}^{\perp}|^2 \mathrm{d}\|V^j\| \right)^{1/2} \\
&\quad \leq C \sup_{B_{1/2}} |D\zeta_a| \sqrt{\tau} E_j,
\end{aligned}$$

where we used the fact that V^j converge in mass to $\mathbf{C}^{(0)}$ and Theorem 3.3.1. Therefore for all j sufficiently large we have

$$\int_{B_{1/2}\setminus G_j} |\nabla^{M^j} x_i \cdot \nabla^{M^j} \zeta_a| \mathrm{d}\|V^j\| \leq C \sup_{B_{1/2}} |D\zeta_a| \sqrt{\tau} E_j. \tag{3.4.20}$$

Next we denote by $\omega_j^{(1)}, \ldots, \omega_j^{(4)}$ the unit vectors in the direction of rays making up the cross section of $\mathbf{C}^j$ and define

$$\begin{aligned}
U_j(\tau) &:= \mathrm{spt}\|\mathbf{C}^j\| \cap B_1 \setminus (B_\tau^{k+1} \times \mathbb{R}^{n-1}), \\
U_j^{(i)}(\tau) &:= B_1 \cap \{(r\omega_j^{(i)}, y) \mid y \in \mathbb{R}^{n-1},\ r > \tau\}.
\end{aligned}$$

We also define

$$\begin{aligned}
G_j(\tau) &:= \mathrm{graph}(u_j|_{U_j(\tau)}), \\
G_j^{(i)}(\tau) &:= \mathrm{graph}(u_j|_{U_j^{(i)}(\tau)}).
\end{aligned}$$

For j large enough, depending on τ, we have

$$|e_{1+k+p}^{\perp}| \geq \frac{1}{2} |D_{y^p} u_j|$$

on $U_j(\tau)$. Indeed we have

$$e_{1+k+p}^{\perp} = e_{1+k+p} - (D_{y^p} u_j \cdot e_{1+k+p}) e_{1+k+p},$$

and hence

$$|e_{1+k+p}^{\perp}| = \frac{|D_{y^p} u_j|}{\sqrt{1+|D_{y^p} u_j|^2}} \geq \frac{1}{2}|D_{y^p} u_j|$$

by Lemma 3.2.6 and Allard's theorem provided j is sufficiently large.

Suppose first of all that $\omega_j^{(1)} = e_1$. Then we can write

$$\nabla^{M^j} x_1 \cdot \nabla^{M^j} \zeta_a = e_1 \cdot \nabla^{M^j} \zeta_a = \sum_{q=1}^{n+k} h^{1q}(X) \partial_q \zeta_a.$$

Now $\partial_p \zeta_a = 0$ for $p = 2, \ldots, k+1$, so

$$\nabla^{M^j} x_1 \cdot \nabla^{M^j} \zeta_a = h^{11} \frac{\partial}{\partial x^1} \zeta_a + \sum_{p=1}^{n-1} h^{1,1+k+p} \frac{\partial}{\partial y^p} \zeta_a.$$

Therefore, invoking the area formula and denoting by h_j and $(h_j^{p,q})$ the determinant and inverse respectively of the $(n-1)\times(n-1)$ matrix $(\delta_{pq} + D_p u_j \cdot D_q u_j)$, it follows that

$$\int_{G_j^{(i)}(\tau)} \nabla^{M^j} x_1 \cdot \nabla^{M^j} \zeta_a \mathrm{d}\|V^j\| = \int_\tau^1 \int_{\mathbb{R}^{n-1}} \left(h_j^{11} \frac{\partial}{\partial x} \zeta_a \left(\sqrt{x^2 + |u_j|^2}, y \right) \right. \\ \left. + \sum_{p=1}^{n-1} h_j^{1,1+k+p} \frac{\partial}{\partial y^p} \zeta_a \left(\sqrt{x^2 + |u_j|^2}, y \right) \right) \sqrt{h_j} \mathrm{d}y \mathrm{d}x.$$

Now by the chain rule we have

$$\zeta_a \left(\sqrt{x^2 + |u_j|^2}, y \right) = \frac{\partial}{\partial y^a} \left(\zeta \left(\sqrt{x^2 + |u_j|^2}, y \right) \right) - \frac{\frac{\partial \zeta}{\partial r} \left(\sqrt{x^2 + |u_j|^2}, y \right) u_j \cdot \frac{\partial u_j}{\partial y^a}}{\sqrt{x^2 + |u_j|^2}},$$

and so by integrating we have

$$\int_0^1 \int_{\mathbb{R}^{n-1}} \frac{\partial^2 \zeta}{\partial r \partial y^a} \left(\sqrt{x^2 + |u_j|^2}, y \right) \mathrm{d}y \mathrm{d}x \\ = \int_0^1 \int_{\mathbb{R}^{n-2}} \left[\frac{\partial \zeta}{\partial r} \left(\sqrt{x^2 + |u_j|^2}, y \right) \right]_{-\infty}^{\infty} \mathrm{d}y \mathrm{d}x = 0.$$

Hence it follows that

$$\int_0^1 \int_{\mathbb{R}^{n-1}} \left(h^{11}(x,y) \frac{\partial}{\partial x} \left(\zeta_a \left(\sqrt{x^2 + |u_j|^2}, y \right) \right) \right) \sqrt{h_j} \mathrm{d}y \mathrm{d}x$$

$$= \int_0^1 \int_{\mathbb{R}^{n-1}} \Bigg(\left(\sqrt{h_j} h_j^{11} - 1 \right) \frac{\partial^2}{\partial y^a \partial x} \left(\zeta \left(\sqrt{x^2 + |u_j|^2}, y \right) \right) - \sqrt{h_j} h_j^{11} \frac{\partial}{\partial x} \left(\frac{\frac{\partial \zeta}{\partial r} \left(\sqrt{x^2 + |u_j|^2}, y \right) u_j \cdot \frac{\partial u_j}{\partial y^a}}{\sqrt{x^2 + |u_j|^2}} \right) \Bigg) \mathrm{d}y \mathrm{d}x.$$

Furthermore, we have that

$$|h_j - 1| \leq C|\nabla u_j|^2,$$
$$(h_j^{pq}) = I - (D_p u_j \cdot D_q u_j) + O(|Du_j|^4),$$

and so it follows

$$\left| \int_{G_j^(1)(\tau)} \nabla^{M^j} x_1 \cdot \nabla^{M^j} \zeta_a \mathrm{d}\|V^j\| \right| \leq C \left(\int_{U_j^{(1)}(\tau/2)} |u_j|^2 \mathrm{d}\mathcal{H}^n \right) \sup_{B_{1/2}} (|D\zeta| + |D^2 \zeta|).$$

Moreover, since $G_j^{(1)}$ is defined by $x_i = u_i(x, 0, y)$ for each $i = 2, \ldots, k+1$, we have for each $i \geq 2$ that

$$\int_{G_j^{(1)}(\tau)} \nabla^{M^j} x_i \cdot \nabla^{M^j} \zeta_a \mathrm{d}\|V^j\| = \int_{U_j^{(1)}(\tau)} \sum_{p,q \neq 2,\ldots,k+1} h_j^{pq} D_p u_j^i D_q \left(\zeta_a \left(\sqrt{x^2 + |u_j|^2}, y \right) \right) \sqrt{h_j} \mathrm{d}\mathcal{H}^n.$$

Now $\sqrt{h_j} \leq 1 + C|\nabla u_j|^2$ and $(h_j^{pq}) = I + A_j$ with $|A_j| \leq C|\nabla u_j|^2$, so it follows that

$$\int_{G_j^{(1)}(\tau)} \sum_{i=1}^{k+1} (e_j \cdot \nabla \zeta) \mathrm{d}\|V^j\| = \int_{U_j^{(1)}(\tau)} \nabla u_j \cdot \nabla \zeta_a \mathrm{d}\mathcal{H}^n + o(E_j).$$

This is invariant under rotations, hence the same formula holds without the assumption $\omega_j^{(1)} = e_1$. Analogous formulae will hold for each $G_j^{(i)}(\tau)$ for $i = 2, 3, 4$. Hence

$$\int_{G_j(\tau)} \sum_{i=1}^{k+1} (e_i \cdot \nabla^{M^j} \zeta_a) \mathrm{d}\|V^j\| = \int_{U_j(\tau)} \nabla u_j \cdot \nabla \zeta_a \mathrm{d}\mathcal{H}^n.$$

Combining this with (3.4.19) we have

$$0 = \int_{B_1} \sum_{i=1}^{k+1} (e_i \cdot \nabla^{M^j} \zeta_a) \mathrm{d}\|V^j\| = \int_{U_j(\tau)} \nabla u_j \cdot \nabla \zeta_a \mathrm{d}\mathcal{H}^n + o(E_j) + S_j E_j,$$

where $|S_j| \leq C\sqrt{\tau}$. Dividing by E_j and passing to the limit $j \to \infty$ and then letting $\tau \searrow 0$ it follows that

$$\int_{\mathbf{C}^{(0)} \cap B_1} \nabla v \cdot \nabla \zeta_a \mathrm{d}\mathcal{H}^n = 0$$

for every ζ with $\partial\zeta/\partial r = 0$ in a neighbourhood of $\{r = 0\}$. As ζ depends only on r and y, we may write the above, after integrating by parts, as

$$\int_H \tilde{v} \Delta \zeta_a \mathrm{d}\mathcal{H}^n = 0,$$

where $H := \{(r, y) \in \mathbb{R}^n \mid r > 0\}$, and $\tilde{v}(r, y) = \sum_{i=1}^4 v(r\omega^{(i)}, y)$, and $\omega^{(i)}$ are the unit vectors in the direction of the 4 rays of the one-dimensional cross section of $\mathbf{C}^{(0)}$. Defining the difference

$$\delta_{a,h}\zeta(r, y) = \zeta(r, y + he_{1+k+a}) - \zeta(r, y),$$

the arbitrariness of ζ implies

$$\int_H \tilde{v}(\delta_{a,h} \Delta \zeta) \mathrm{d}\mathcal{H}^n = 0$$

for every $|h| < 1/16$ and for $\zeta \in C_c^\infty(B_{5/16}^n)$, because such ζ satisfies $\delta_{a,h}\zeta \in C_c^\infty(B_{3/8}^n)$ and $\partial(\delta_{a,h}\zeta)/\partial r \equiv 0$ in a neighbourhood of $\{r = 0\}$. After a change of variables we find

$$\int_H (\delta_{a,h}\tilde{v}) \Delta \zeta \mathrm{d}\mathcal{H}^n = 0 \tag{3.4.21}$$

for all $|h| < 1/16$. It is easy to see that (3.4.21) holds for and $\zeta(r, y)$ that is even in the r variable can be approximated in C^2_{loc} by a sequence ζ_l with $\partial\zeta_l/\partial r \equiv 0$ in a neighbourhood of $\{r = 0\}$. Therefore, if $\hat{v}$ is the even extension of $\tilde{v}$, we have

$$\int_{B_{5/16}^n} (\delta_{a,h}\hat{v}) \Delta \zeta \mathrm{d}\mathcal{H}^n = 0. \tag{3.4.22}$$

Moreover, we trivially have the same identity if $\zeta(r, y)$ is odd in the r variable. Thus (3.4.22) holds for any $\zeta \in C_c^\infty(B_{5/16}^n)$, so by Weyl's Lemma $\delta_{a,h}\hat{v}$ is a smooth harmonic function. Now by a change of variables, we have that

$$\left| \int_{B_{9/32}^n} \delta_{a,h}\hat{v} \mathrm{d}\mathcal{H}^n \right| = \left| \int_{B_{9/32}^n(h)} \hat{v} \mathrm{d}\mathcal{H}^n - \int_{B_{9/32}^n} \hat{v} \mathrm{d}\mathcal{H}^n \right| \leq C|h| \sup_{B_{5/16}^n} |\hat{v}|,$$

provided $|h| \leq 1/32$. Consequently, it follows from standard estimates for harmonic functions, that there is a harmonic function $\hat{v}_a\colon B^n_{17/64} \to \mathbb{R}^k$ such that $h^{-1}\delta_{a,h}\hat{v} \to \hat{v}_a$ in $C^2(B^n_{17/64})$ as $h \to 0$ with the estimate

$$\sup_{B^n_{17/64}} \left(|\hat{v}_a|^2 + |D\hat{v}_a|^2 + |D^2\hat{v}_a|^2\right) \leq C \int_{B_{1/2}} |v|^2 \mathrm{d}\mathcal{H}^n.$$

Since $\hat{v}_a = \partial\hat{v}/\partial y_a$ on $B^n_{17/64} \setminus B$, we have that

$$\hat{v}(x,y) = \hat{v}(x, y_1, \ldots, 0, \ldots, y_{n-1}) + \int_0^{y_a} \hat{v}_a(x, y_1, \ldots, t, \ldots, y_{n-1})\mathrm{d}t,$$

and hence letting $x \to 0$ on each side it follows that with $Y = (0, y)$ we have

$$\frac{\partial\hat{v}}{\partial y_a}(Y) = \hat{v}_a(Y), \quad \frac{\partial^2\hat{v}}{\partial y_m \partial y_a}(Y) = \frac{\partial\hat{v}_a}{\partial y_m}(Y), \quad \frac{\partial^3\hat{v}}{\partial y_l \partial y_m \partial y_a}(Y) = \frac{\partial^2\hat{v}_a}{\partial y_l \partial y_m}(Y),$$

and that $\hat{v}$ satisfies the estimate

$$\sup_{B^n_{17/64}} \left(|\hat{v}|^2 + |D_Y\hat{v}|^2 + |D^2_Y\hat{v}|^2 + |D^3_Y\hat{v}|^2\right) \leq C \int_{B_{1/2}} |v|^2 \mathrm{d}\mathcal{H}^n.$$

Now $\hat{v}(Y) = \sum_{i=1}^4 \kappa_i(Y)$, where $\kappa_i(Y) \in \mathbb{R}^k$ is the result of projecting $\kappa(Y)$ to $(\omega^{(i)})^\perp$ in $\mathbb{R}^{k+1}$ and then identifying $(\omega^{(i)})^\perp$ with $\mathbb{R}^k$. Because of the radial symmetry of ζ in (3.4.22), we have freedom to choose this identification. Moreover the normal spaces to each of the $\omega^{(i)}$ are k-dimensional and don't coincide, so they span $\mathbb{R}^{k+1}$. Hence $\kappa \in C^\infty(B \cap B_{17/64}; B^\perp)$ and we have

$$\sup_{B \cap B_{17/64}} \left(|\kappa|^2 + |D_Y\kappa|^2 + |D^2_Y\kappa|^2 + |D^3_Y\kappa|^2\right) \leq C \int_{B_{1/2}} |v|^2 \mathrm{d}\mathcal{H}^n.$$

Combining this with Lemma 3.4.4 and the standard boundary regularity theory for harmonic functions, see for example [22], the desired estimates follow. □

3.5 Excess decay

Here we prove the main excess decay lemma. This is based on [52, Lemma 1], and follows from a blow-up argument using the regularity properties of blow-ups from the previous section.

Lemma 3.5.1 (Excess decay lemma)**.** *Let $\theta \in (0, 1/4)$. There exists $\varepsilon_0 = \varepsilon_0(\mathbf{C}^{(0)}, \theta)$ such that if $V \in \mathcal{V}$, $\mathbf{C} \in \mathcal{C}$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_0$ and $\delta_A = 1/16$, then there is a $\tilde{\mathbf{C}} \in \mathcal{C}$ and a rotation Γ such that*

$$
\begin{aligned}
|\Gamma - \mathrm{id}| &\le CE_V(\mathbf{C}), \qquad (3.5.1)\\
\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\tilde{\mathbf{C}}\| \cap B_1,\ \mathrm{spt}\|\mathbf{C}\| \cap B_1) &\le CE_V(\mathbf{C}),\\
\frac{1}{\theta^{n+2}} \int_{\Gamma(B_{\theta/R_0} \setminus (B^{k+1}_{\theta/4R_0} \times \mathbb{R}^{n-1}))} \mathrm{dist}^2(X, \mathrm{spt}\|V\|)\mathrm{d}\|\Gamma_\# \tilde{\mathbf{C}}\| &\\
+ \frac{1}{\theta^{n+2}} \int_{B_\theta} \mathrm{dist}^2(X, \mathrm{spt}\|\Gamma_\# \tilde{\mathbf{C}}\|)\mathrm{d}\|V\| &\le C\theta^2 E_V^2(\mathbf{C}),
\end{aligned}
$$

where $C = C(\mathbf{C}^{(0)})$ and $\gamma = \gamma(\mathbf{C}^{(0)}, \theta) \ge 1$.

Proof. Fix $\theta \in (0, 1/4)$ and take sequences $\varepsilon_j \searrow 0$, $V^j \in \mathcal{V}$ and $\mathbf{C}^j \in \mathcal{C}$ such that V^j, $\mathbf{C}^j$ and $\mathbf{C}^{(0)}$ satisfy Hypotheses A with $\varepsilon_A = \varepsilon_j$ and $\delta_A = 1/16$. Define $E_j := E_{V^j}$. We seek to prove that the conclusions of the lemma hold for infinitely many j along this sequence. For each $i = 1, \ldots, n-1$ we let $Y_i := \frac{\theta}{2} e_{k+1+i} \in B$. Lemma 3.4.1 implies that there exist sequences $Z_{i,j} \in \mathrm{spt}\|V^j\| \cap B_1$ such that $\Theta(\|V^j\|, Z_{i,j}) \ge 2$ and $Z_{i,j} \to Y_i$ as $j \to \infty$.

For large j, the $Z_{i,j}$ must span an $(n-1)$-dimensional subspace Σ_j of $\mathbb{R}^{n+k}$. We choose the rotations Γ_j such that $\Gamma_j(\Sigma_j) = B$ and Γ_j minimises $|\Gamma - \mathrm{id}|$ among all Γ which align Σ_j with B. Since Lemma 3.3.5 implies that $\mathrm{dist}^2(Z_{i,j}, B) \le CE_j^2$ for each i, it follows that

$$|\Gamma_j - \mathrm{id}| \le CE_j,$$

for j sufficiently large. Thus

$$\mathrm{dist}_{\mathcal{H}}(\Gamma_j^{-1}(B) \cap B_1,\ B \cap B_1) \le CE_j,$$

and so by the triangle inequality

$$\tilde{E}_j^2 := \int_{B_1} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}^j\|)\mathrm{d}\|\Gamma_{j\#} V^j\| \le CE_j^2.$$

Denote by $\tilde{v}$ the blow-up of $\Gamma_{j\#} V^j$ off $\mathbf{C}^{(0)}$ relative to $\mathbf{C}^j$. By construction, we have $\tilde{v}(Y_i) = 0$ for $i = 1, \ldots, n-1$, and so for each $i = 1, \ldots, n-1$ and each

$l = 1, \dots, 4$ there exists $S_{l,i} \in B_{\theta/2} \cap (\{0\}^{k+1} \times \mathbb{R}^{n-1})$ such that

$$\frac{\partial \tilde{v}_l}{\partial y_i}(S_{l,i}) = 0,$$

where $\tilde{v}_l$ denotes the C^2 extension to B of $\tilde{v}|_{H_l^{(0)}}$. Hence by Theorem 3.4.5 it follows that

$$|D_y \tilde{v}_l(0)|^2 \leq C\theta^2 \int_{B_{1/2} \cap \mathbf{C}^{(0)}} |\tilde{v}|^2 \mathrm{d}\mathcal{H}^n, \tag{3.5.2}$$

for each $l = 1, \dots, 4$. Suppose that $H_1^{(0)} = [0, \infty) \times \{0\}^k \times \mathbb{R}^{n-1}$. We define two linear functions on $H_1^{(0)}$ as follows

$$p_1 \colon H_1^{(0)} \to (H_1^{(0)})^\perp, \ \ (r, y) \mapsto r\frac{\partial \tilde{v}}{\partial r}(0) + D_y\tilde{v}(0) \cdot y = D\tilde{v}(0) \cdot (r, y),$$
$$c_1 \colon H_1^{(0)} \to (H_1^{(0)})^\perp, \ \ (r, y) \mapsto r\frac{\partial \tilde{v}}{\partial r}(0).$$

Hence the graph of p_1 over $H^{(0)}$ contains the tangent half-plane to $\tilde{v}_1$ at the origin, while the graph of c_1 over $H_1^{(0)}$ is a half-plane with boundary equal to B. Moreover, it follows from the definitions and the estimate (3.5.2) that

$$|p_1 - c_1|^2 \leq C\theta^2 |y|^2 \int_{B_{1/2} \cap \mathbf{C}^{(0)}} |\tilde{v}|^2 \mathrm{d}\mathcal{H}^n.$$

This estimate is invariant under rotations, and so holds without our assumption on $H_1^{(0)}$. By the same reasoning, analogous estimates hold for each $H_l^{(0)}$ where $l = 2, \dots, 4$. Combining this with Theorem 3.4.5 we find

$$\sum_{l=1}^{4} \frac{1}{\theta^{n+2}} \int_{B_{2\theta} \cap H_l^{(0)}} |\tilde{v}_l - c_l|^2 \mathrm{d}\mathcal{H}^n \leq C\theta^2, \tag{3.5.3}$$

where c_l is defined analogously to c_1 for $l = 2, 3, 4$. We now define the function $c \colon \operatorname{spt}\|\mathbf{C}^{(0)}\| \to (\operatorname{spt}\|\mathbf{C}^{(0)}\|)^\perp$ by insisting that $c|_{H_l^0} = c_l$. Then we construct a new sequence of cones $\tilde{\mathbf{C}}^j$ as follows. If each $\mathbf{C}^j$ is the graph of some ψ_j over $\mathbf{C}^{(0)}$, then we define $\tilde{\psi}_j := \psi_j + E_j c$. It then follows that if u_j and $\tilde{u}_j$ denote the graph functions obtained from Lemma 3.2.6 applied to $\Gamma_{j\#}V^j$ and $\mathbf{C}^j$, and $\Gamma_{j\#}V^j$ and $\tilde{\mathbf{C}}^j$ respectively, then they satisfy a relation of the form

$$\tilde{u}_j(X + \tilde{\psi}_j(X)) = u_j(X + \psi_j(X)) - E_j c + o(E_j).$$

Hence it follows from (3.4.2) and (3.5.3) that

$$\lim_{j\to\infty}\frac{1}{\theta^{n+2}E_j^2}\int_{B_\theta}\mathrm{dist}^2(X,\mathrm{spt}\|\tilde{\mathbf{C}}^j\|)\mathrm{d}\|\Gamma_{j\#}V^j\| = \frac{1}{\theta^{n+2}}\int_{B_\theta}|\tilde{v}-c|^2\mathrm{d}\mathcal{H}^n \le C\theta^2,$$

and so for j large

$$\frac{1}{\theta^{n+2}}\int_{B_\theta}\mathrm{dist}^2(X,\mathrm{spt}\|(\Gamma_j^{-1})_\#\tilde{\mathbf{C}}^j\|)\mathrm{d}\|V^j\| \le C\theta^2E_j^2. \tag{3.5.4}$$

Since $\mathrm{spt}\|V^j\|\cap B_{\theta/R_0}\setminus(B^{k+1}_{\theta/4R_0}\times\mathbb{R}^{n-1})$ coincides with the graph of a smooth single-valued function on $\Gamma_{j\#}\tilde{\mathbf{C}}^j$, provided j is sufficiently large (depending on θ) we have

$$\begin{aligned}\frac{1}{\theta^{n+2}}\int_{B_{\theta/R_0}\setminus(B^{k+1}_{\theta/4R_0}\times\mathbb{R}^{n-1})}&\mathrm{dist}^2(X,\mathrm{spt}\|V^j\|)\mathrm{d}\|\Gamma_{j\#}\tilde{\mathbf{C}}^j\|\\ &\le\frac{C}{\theta^{n+2}}\int_{B_\theta}\mathrm{dist}^2(X,\mathrm{spt}\|\Gamma_{j\#}\tilde{\mathbf{C}}^j\|)\mathrm{d}\|V^j\|.\end{aligned}$$

Moreover, the definition of $\tilde{\mathbf{C}}^j$ clearly implies

$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\tilde{\mathbf{C}}^j\|\cap B_1,\ \mathrm{spt}\|\mathbf{C}^j\|\cap B_1)\le CE_j,$$

and so the result follows. □

3.6 Regularity theorems

With Lemma 3.5.1 in hand, we can now prove the following regularity theorem.

Theorem 3.6.1 (Regularity theorem). *There are constants $\varepsilon=\varepsilon(\mathbf{C}^{(0)})>0$ and $\alpha=\alpha(\mathbf{C}^{(0)})\in(0,1)$ such that if $V\in\mathcal{V}$, V is stationary in B_{R_0}, and*

$$\frac{\|V\|(B_{R_0})}{\omega_nR_0^n}\le 2+\frac{1}{32},\qquad Q_V(\mathbf{C}^{(0)})<\varepsilon,$$

then the following conclusions hold.

(1) There is a $C^{1,\alpha}$ function $w\colon B\to B^\perp$ with $\|w\|_{1,\alpha}\le CQ_V(\mathbf{C}^{(0)})$ and such that $\mathrm{sing}V\cap B_{1/2}=\mathrm{graph}(w)\cap B_{1/2}$.

(2) There are smooth embedded n-dimensional minimal submanifolds M_i for $i = 1, \ldots, 4$ such that $\partial M_i \cap B_{1/2} = \mathrm{graph}(w) \cap B_{1/2}$ for each $i = 1, \ldots, 4$.

(3) At every $Z \in \mathrm{sing}V \cap B_{1/2}$ there exists a unique tangent cone $\mathbf{C}_Z$, which consists of four half-planes meeting along a common $(n-1)$-dimensional subspace, and satisfies the decay estimate

$$\frac{1}{\rho^{n+2}} \int_{B_\rho(Z)} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}_Z\|) \mathrm{d}\|V\| \leq C\rho^{2\alpha} Q_V^2(\mathbf{C}^{(0)}),$$

for each $\rho \in (0, 1/4]$ and where $C = C(\mathbf{C}^{(0)})$.

Proof. Pick $\theta \in (0, 1/4)$ such that $C\theta^2 R_0^{n+2} < 1/4$, where C is the constant from Lemma 3.5.1. We claim that if ε is chosen small enough, then by iterating Lemma 3.5.1 we can produce sequences of rotations Γ_j and cones $\mathbf{C}^j \in \mathcal{C}$ such that

(A)

$$|\Gamma_j - \Gamma_{j-1}| \leq \frac{C}{2^j} Q_V(\mathbf{C}^{(0)})$$

where we define $\Gamma_0 := \mathrm{id}$,

(B)

$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\mathbf{C}^j\| \cap B_1, \mathrm{spt}\|\mathbf{C}^{j-1}\| \cap B_1) \leq \frac{C}{2^j} Q_V(\mathbf{C}^{(0)})$$

where we define $\mathbf{C}^0 := \mathbf{C}^{(0)}$,

(C)

$$\frac{R_0^{j(n+2)}}{\theta^{j(n+2)}} \int_{B_{\theta^j R_0^{-(j-1)}}} \mathrm{dist}^2(X, \mathrm{spt}\|\Gamma_{j\#}\mathbf{C}^j\|) \mathrm{d}\|V\| \leq \frac{1}{4^j} Q_V^2(\mathbf{C}^{(0)}), \quad \text{and}$$

(D)

$$\frac{R_0^{j(n+2)}}{\theta^{j(n+2)}} \int_{\Gamma_j(B_{\theta^j R_0^{-j}} \setminus (B^{k+1}_{\theta^j/(4R_0^j)} \times \mathbb{R}^{n-1}))} \mathrm{dist}^2(X, \mathrm{spt}\|V\|) \mathrm{d}\|\Gamma_{j\#}\mathbf{C}^j\| \leq \frac{1}{4^j} Q_V^2(\mathbf{C}^{(0)}).$$

We prove this by induction. Suppose first that ε_0 is as in Lemma 3.5.1, then provided $\varepsilon < \varepsilon_0$ we may apply Lemma 3.5.1 with $\mathbf{C} = \mathbf{C}^{(0)}$, to conclude that there exist $\mathbf{C}^1 \in \mathcal{C}$ and a rotation Γ_1 such that

$$|\Gamma_1 - \mathrm{id}| \leq CE_V(\mathbf{C}^{(0)}),$$
$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\mathbf{C}^1\| \cap B_1,\ \mathrm{spt}\|\mathbf{C}^{(0)}\| \cap B_1) \leq CE_V(\mathbf{C}^{(0)}),$$
$$\frac{1}{\theta^{n+2}} \int_{B_\theta} \mathrm{dist}^2(X, \mathrm{spt}\|\Gamma_{1\#}\mathbf{C}_1\|)\mathrm{d}\|V\| \leq C\theta^2 E_V^2(\mathbf{C}^{(0)}) \leq \frac{1}{4R_0^{n+2}} E_V^2(\mathbf{C}^{(0)}),$$
$$\frac{1}{\theta^{n+2}} \int_{\Gamma_1(B_{\theta R_0^{-1}} \setminus (B^{k+1}_{\theta(4R_0)^{-1}} \times \mathbb{R}^{n-1}))} \mathrm{dist}^2(X, \mathrm{spt}\|V\|)\mathrm{d}\|\Gamma_{1\#}\mathbf{C}^1\| \leq \frac{1}{4R_0^{n+2}} E_V^2(\mathbf{C}^{(0)}).$$

Since $E_V^2(\mathbf{C}^{(0)}) \leq Q_V^2(\mathbf{C}^{(0)})$ the base case evidently follows. Now suppose that we have found some sequences $\mathbf{C}^1, \ldots, \mathbf{C}^j$ and $\Gamma_1, \ldots, \Gamma_j$ satisfying (A)-(D) and we wish to construct $\mathbf{C}^{j+1}$ and Γ_{j+1}. We do this by applying Lemma 3.5.1 again, with $\mathbf{C}^j$ in place of $\mathbf{C}$, and $V^j := (\eta_{0,\theta^j R_0^{-j}} \circ \Gamma_j^{-1})_\# V$ in place of V. To do so we need to check that we can choose ε small, independently of j, to ensure both $Q_{V^j}(\mathbf{C}^{(0)}) < \varepsilon_0$ and $\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\mathbf{C}^j\| \cap B_1,\ \mathrm{spt}\|\mathbf{C}^{(0)}\|) \leq \varepsilon_0$.

Notice first that (B) and the triangle inequality together imply that

$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\mathbf{C}^j\| \cap B_1,\ \mathrm{spt}\|\mathbf{C}^{(0)}\| \cap B_1) \leq C\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^j}\right) Q_V(\mathbf{C}^{(0)}) \leq C\varepsilon, \tag{3.6.1}$$

where $C = C(\mathbf{C}^{(0)})$ is independent of j. Therefore we need only choose ε small enough to ensure $C\varepsilon < \varepsilon_0$.

We next show that $Q_{V^j}(\mathbf{C}^{(0)}) < \varepsilon_0$. We may assume that $Q_{V^{j-1}}(\mathbf{C}^{(0)}) < \varepsilon_0$, since this is the case by assumption for $j = 1$, and we will establish the same for $j = j + 1$ presently. Combining (A) with the triangle inequality yields

$$|\Gamma_j - \mathrm{id}| \leq C\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^j}\right) Q_V(\mathbf{C}^{(0)}) \leq C\varepsilon, \tag{3.6.2}$$

where $C = C(\mathbf{C}^{(0)})$ is independent of j. Furthermore, (C) and (D) together imply

$$Q_{V^j}^2(\mathbf{C}^j) \leq \frac{2}{4^j} Q_V^2(\mathbf{C}^{(0)}). \tag{3.6.3}$$

It therefore follows from (3.6.3), the triangle inequality and (3.6.1) that

$$E_{V^j}^2(\mathbf{C}^{(0)}) = \int_{B_1} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}^{(0)}\|)\mathrm{d}\|V^j\| \leq CQ_V^2(\mathbf{C}^{(0)}).$$

Since $Q_{V^{j-1}}(\mathbf{C}^{(0)}) < \varepsilon_0$, we can apply Lemma 3.2.5 to V^{j-1} and, supposing that ε_0

is sufficiently small, deduce from (3.6.2) that V^j is graphical with small gradient over $\mathbf{C}^{(0)}$ in the region $B_1 \setminus (B^{k+1}_{1/4} \times \mathbb{R}^{n-1})$. It follows that

$$\int_{B_1\setminus(B^{k+1}_{1/4}\times\mathbb{R}^{n-1}))} \mathrm{dist}^2(X, \mathrm{spt}\|V^j\|)\mathrm{d}\|\mathbf{C}^{(0)}\| \le CE^2_{V^j}(\mathbf{C}^{(0)}) \le CQ^2_V(\mathbf{C}^{(0)}),$$

where C is again an absolute constant depending only on $\mathbf{C}^{(0)}$, not on j. Piecing all of this together, we can find a constant $C = C(\mathbf{C}^{(0)})$ such that

$$\begin{aligned} Q^2_{V^j}(\mathbf{C}^{(0)}) &\le CQ^2_V(\mathbf{C}^{(0)}), \\ \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\mathbf{C}^j\| \cap B_1,\ \mathrm{spt}\|\mathbf{C}^{(0)}\| \cap B_1) &\le CQ_V(\mathbf{C}^{(0)}). \end{aligned}$$

We choose ε small enough that $C\varepsilon < \varepsilon_0$, which allows us to apply Lemma 3.5.1 to V^j and $\mathbf{C}^j$. This produces a rotation Γ and a cone $\mathbf{C}^{j+1} \in \mathcal{C}$ such that

$$\begin{aligned} |\Gamma - \mathrm{id}| &\le CE_{V^j}(\mathbf{C}^j), \\ \mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\mathbf{C}^{j+1}\| \cap B_1,\ \mathrm{spt}\|\mathbf{C}^j\| \cap B_1) &\le CE_{V^j}(\mathbf{C}^j), \\ \frac{1}{\theta^{n+2}} \int_{B_\theta} \mathrm{dist}^2(X, \mathrm{spt}\|\Gamma_\#\mathbf{C}^{j+1}\|)\mathrm{d}\|V^j\| &\le C\theta^2 E^2_{V^j}(\mathbf{C}^j), \\ \frac{1}{\theta^{n+2}} \int_{\Gamma(B_{\theta R_0^{-1}}\setminus(B^{k+1}_{\theta(4R_0)^{-1}}\times\mathbb{R}^{n-1}))} \mathrm{dist}^2(X, \mathrm{spt}\|V^j\|)\mathrm{d}\|\Gamma_\#\mathbf{C}^{j+1}\| &\le C\theta^2 E^2_{V^j}(\mathbf{C}^j). \end{aligned}$$

Defining $\Gamma_{j+1} := \Gamma\circ\Gamma_j$, and noting that (C) implies that $E^2_{V^j}(\mathbf{C}^j) \le 4^{-j}Q^2_V(\mathbf{C}^{(0)})$, properties (A)-(D) clearly follow for $\mathbf{C}^{j+1}$ and Γ_{j+1}, thus establishing (A)-(D) for all $j \ge 1$ by induction.

Next observe that given any $Z \in \mathrm{sing}V \cap B_{1/2}$, Lemma 3.3.5 implies we can apply the above reasoning to $V_Z := \eta_{Z,1-(2R_0)^{-1}\#}V$, provided we choose ε small enough (but independent of Z). From properties (A) and (B) we deduce the existence of sequences of rotations $\Gamma_{Z,j}$ and cones $\mathbf{C}^j_Z \in \mathcal{C}$ with $\Gamma_{Z,j} \to \Gamma_Z$ and $\mathbf{C}^j_Z \to \mathbf{C}_Z \in \mathcal{C}$. Moreover the following properties hold.

(I)
$$|\Gamma_Z - \mathrm{id}| \le CQ_{V_Z}(\mathbf{C}^{(0)}).$$

(II)
$$\mathrm{dist}_{\mathcal{H}}(\mathrm{spt}\|\mathbf{C}_Z\| \cap B_1,\ \mathrm{spt}\|\mathbf{C}^{(0)}\| \cap B_1) \le CQ_{V_Z}(\mathbf{C}^{(0)}).$$

(III) There is $\alpha = \alpha(\mathbf{C}^{(0)})$ such that for each $\rho \in (0, \theta)$ we have

$$\frac{1}{\rho^{n+2}} \int_{\Gamma_Z(B_{\rho R_0^{-1}} \setminus (B^{k+1}_{\rho(4R_0)^{-1}} \times \mathbb{R}^{n-1}))} \mathrm{dist}^2(X, \mathrm{spt}\|V_Z\|) \mathrm{d}\|\Gamma_{Z\#}\mathbf{C}_Z\|$$
$$\leq C\rho^{2\alpha} Q^2_{V_Z}(\mathbf{C}^{(0)}).$$

(IV) For α as in (III) we have

$$+\frac{1}{\rho^{n+2}} \int_{B_\rho} \mathrm{dist}^2(X, \mathrm{spt}\|\Gamma_{Z\#}\mathbf{C}_Z\|) \mathrm{d}\|V_Z\| \leq C\rho^{2\alpha} Q^2_{V_Z}(\mathbf{C}^{(0)}).$$

Notice in particular that (III) and (IV) imply that $\Gamma_{Z\#}\mathbf{C}_Z$ is the unique tangent cone to V at Z. Properties (III) and (IV) follow by a similar argument to that used in the proof of Lemma 3.4.4, namely by interpolating the scales θ^j and then choosing α such that $\theta^{2\alpha} = 1/4$.

Let $y \in B^{n-1}_{1/2}(0)$ and suppose that $\mathrm{sing} V \cap B_{1/2} \cap (\mathbb{R}^{k+1} \times \{y\})$ contains more than one point. Choose any two such points Z_1 and Z_2, and define $\sigma := |Z_1 - Z_2| > 0$. Suppose that at least one of the Z_i has $\Theta(\|V\|, Z_i) \geq 2$, indeed suppose without loss of generality that $\Theta(\|V\|, Z_1) \geq 2$. Note that Lemma 3.3.5 implies that $\sigma < \theta$ provided ε is suitably small, so by properties (III) and (IV) applied at Z_1 we have

$$\frac{1}{(2\sigma)^{n+2}} \int_{\Gamma_{Z_1}(B_{2\sigma/R_0} \setminus (B^{k+1}_{2\sigma/(4R_0)} \times \mathbb{R}^{n-1}))} \mathrm{dist}^2(X, \mathrm{spt}\|V_{Z_1}\|) \mathrm{d}\|\Gamma_{Z_1\#}\mathbf{C}_{Z_1}\|$$
$$\leq C(2\sigma)^{2\alpha} Q^2_{V_{Z_1}}(\mathbf{C}^{(0)})$$
$$\frac{1}{(2\sigma)^{n+2}} \int_{B_{2\sigma}} \mathrm{dist}^2(X, \mathrm{spt}\|\Gamma_{Z_1\#}\mathbf{C}_{Z_1}\|) \mathrm{d}\|V_{Z_1}\| \leq C(2\sigma)^{2\alpha} Q^2_{V_{Z_1}}(\mathbf{C}^{(0)}).$$

Rescaling this implies

$$Q^2_{\eta_{0,2\sigma/R_0\#}V_{Z_1}}(\mathbf{C}^{(0)}) \leq C Q^2_{V_{Z_1}}(\mathbf{C}^{(0)}),$$

with $C = C(\mathbf{C}^{(0)})$ independent of σ. Assuming again that ε was initially small enough, we conclude from Lemma 3.2.5 that $\eta_{0,2\sigma/R_0\#}V_{Z_1}$ is smooth inside $B_1 \setminus (B^{k+1}_{1/4} \times \mathbb{R}^{n-1})$, but this is a contradiction, as $\eta_{0,2\sigma/R_0\#}V_{Z_1}$ has a singularity on $\partial B^{k+1}_{1/2}(0) \times \{0\}^{n-1}$ by construction. Consequently, whenever a slice $\mathrm{sing} V \cap (\mathbb{R}^{k+1} \times \{y\}) \cap B_{1/2}$ for some $y \in B^{n-1}_{1/2}$ contains at least one point Z of density greater than or equal to 2, then in fact $\operatorname{sing} V \cap (\mathbb{R}^{k+1} \times \{y\}) \cap B_{1/2} = \{Z\}$. It follows that $\{Z \in B_{1/2} \mid \Theta(\|V\|, Z) \geq 2\}$ is contained in the graph of some function $w \colon B \to B^{\perp}$.

Next observe that by (III) and (IV), it follows that if $\rho \in (0, \theta)$ and we define $\tilde{V}_Z := (\eta_{0,\rho R_0^{-1}} \circ \Gamma_Z^{-1})_{\#} V_Z$, then

$$Q^2_{\tilde{V}_Z}(\mathbf{C}_Z) \leq C\rho^{2\alpha} Q^2_{V_Z}(\mathbf{C}^{(0)}).$$

Hence, we can apply Theorem 3.3.1 to conclude that for every $\tilde{Y} \in \operatorname{sing}\tilde{V}_Z \cap B_{1/2}$ with $\Theta(\|\tilde{V}_Z\|, \tilde{Y}) \geq 2$ we have

$$\operatorname{dist}^2(\tilde{Y}, B) \leq Q^2_{\tilde{V}_Z}(\mathbf{C}_Z),$$

and so

$$\rho^{-1-\alpha}\operatorname{dist}(Y, \Gamma_Z(B)) \leq C\varepsilon \tag{3.6.4}$$

for every $Y \in \operatorname{sing} V_Z \cap B_{\rho/2}$ with $\Theta(\|V_Z\|, Y) \geq 2$. Since we also know that $|\Gamma_Z - \mathrm{id}| \leq C\varepsilon$, it follows that w is Lipschitz continuous with constant at most 1 say, provided that ε is small enough. Lemma 3.4.1 implies that $\mathcal{H}^{n-1}$-almost every slice $B_{1/2} \cap (\mathbb{R}^{k+1} \times \{y\})$ contains a singularity Z with density at least 2, hence good density points form a dense subset of $\operatorname{graph}(w)$. Since the singular set is closed and the density is upper semi-continuous, it follows that $\{Y \in \operatorname{sing} V \mid \Theta(\|V\|, Y) \geq 2\} \cap B_{1/2} = \operatorname{graph}(w) \cap B_{1/2}$, and hence by the earlier argument that $\operatorname{sing} V \cap B_{1/2} = \operatorname{graph}(w) \cap B_{1/2}$.

It only remains to show that w is $C^{1,\alpha}$ with the claimed estimate. Notice first that it follows from (3.6.4) that w is differentiable at every $z \in B$, and that if $Z = w(z)$ denotes the corresponding singularity $Z \in \operatorname{sing} V \cap B_{1/2}$ then the tangent plane to $\operatorname{graph}(w)$ at Z is $\Gamma_Z(B)$.

Next observe that given Z_1, $Z_2 \in \operatorname{sing} V \cap B_{1/2}$, by setting $\sigma = R_0|Z_1 - Z_2|$, and provided that $4\sigma R_0^{-1} < \theta$, it follows from property (III) that $Q^2_{\hat{V}}(\mathbf{C}_{Z_1}) \leq C\sigma^{2\alpha} Q^2_V(\mathbf{C}^{(0)})$, where $\hat{V} := (\eta_{0,4\sigma R_0^{-1}} \circ \Gamma_{Z_1}^{-1})_{\#} V_{Z_1}$. We then repeat the previous iteration scheme that established properties (I)-(IV), with $\hat{V}$ in place of V, $\mathbf{C}_{Z_1}$ in place of $\mathbf{C}^{(0)}$ and $\hat{Z} := R_0(4\sigma)^{-1}\Gamma_{Z_1}^{-1}(Z_2 - Z_1)$ in place of Z. The conclusion is the existence of a rotation $\hat{\Gamma}_{\hat{Z}}$ and a cone $\hat{C}_{\hat{Z}}$ such that

$$|\hat{\Gamma}_{\hat{Z}} - \mathrm{id}| \leq CQ_{\hat{V}}(\mathbf{C}_{Z_1}),$$

$$\text{dist}_{\mathcal{H}}(\text{spt}\|\hat{\mathbf{C}}_{\hat{Z}}\| \cap B_1, \text{spt}\|\mathbf{C}_{Z_1}\| \cap B_1) \leq CQ_{\hat{V}}(\mathbf{C}_{Z_1}),$$

$$\frac{1}{\rho^{n+2}} \int_{\hat{\Gamma}_{\hat{Z}}(B_{\rho R_0^{-1}} \setminus (B^{k+1}_{\rho(4R_0)^{-1}} \times \mathbb{R}^{n-1}))} \text{dist}^2(X, \text{spt}\|\hat{V}\|) \mathrm{d}\|\hat{\Gamma}_{\hat{V}\#}\hat{\mathbf{C}}_{\hat{Z}}\| \leq C\rho^{2\alpha} Q^2_{\hat{V}}(\hat{\mathbf{C}}_{\hat{Z}}),$$

$$\frac{1}{\rho^{n+2}} \int_{B_\rho} \text{dist}^2(X, \text{spt}\|\hat{\Gamma}_{\hat{Z}\#}\hat{\mathbf{C}}_{\hat{Z}}\|) \mathrm{d}\|\hat{V}\| \leq C\rho^{2\alpha} Q^2_{\hat{V}}(\hat{\mathbf{C}}_{\hat{Z}}).$$

It follows that $\hat{\Gamma}_{\hat{Z}\#}\hat{\mathbf{C}}_{\hat{Z}}$ is the unique tangent cone to $\hat{V}$ at $\hat{Z}$. However we also know that $\Gamma_{Z_2\#}\mathbf{C}_{Z_2}$ is the unique tangent cone to V at Z_2, and so we deduce that $\hat{\mathbf{C}}_{\hat{Z}} = \mathbf{C}_{Z_2}$ and that $\hat{\Gamma}_{\hat{Z}} = \Gamma_{Z_2} \circ \Gamma_{Z_1}^{-1}$. It therefore follows that

$$|\Gamma_{Z_2} - \Gamma_{Z_1}| = |\Gamma_{Z_2} \circ \Gamma_{Z_1}^{-1} - \text{id}| \leq C\sigma^\alpha Q_V(\mathbf{C}^{(0)}) = C|Z_1 - Z_2|^\alpha Q_V(\mathbf{C}^{(0)})$$

provided that Z_1 and Z_2 were sufficiently close to begin with, but the inequality holds trivially if $|Z_1 - Z_2| \geq \theta/4$ by the triangle inequality and the fact that $|\Gamma_{Z_1} - \text{id}| \leq CQ_V(\mathbf{C}^{(0)})$. Hence the tangent planes of w vary Hölder continuously, implying that w is $C^{1,\alpha}$ and we have the estimate

$$\|w\|_{1,\alpha} \leq CQ_V(\mathbf{C}^{(0)}).$$

□

The above theorem combined with Allard's reflection principle [2] (see also Lemma 1.2.8) implies the following boundary regularity result for a subspace boundary. Before stating it, we define the boundary singular set.

Definition 3.6.2 (Boundary singular set)**.** *Let $U \subset \mathbb{R}^{n+k}$ be open, and let $B \subset U$ be an $(n-1)$-dimensional C^1 submanifold. Let V be an integral n-varifold that is stationary in $U \setminus B$. We define the* boundary singular set, *denoted* $\text{sing}_B V$ *to be the set of all $x \in B$, such that there is no neighbourhood $W \subset U$ of x for which $M := \text{spt}\|V\| \cap W$ consists of a smooth n-dimensional submanifold of W with $\partial M \cap W = B \cap W$.*

Remark 3.6.3. *Notice that the boundary regularity theorem of Allard and Bourni implies that there is an $\varepsilon > 0$ such that*

$$\text{sing}_B V = \{x \in B \mid \Theta(\|V\|, x) \geq 1/2 + \varepsilon\}.$$

Corollary 3.6.4. *There exists $\varepsilon = \varepsilon(\mathbf{C}^{(0)}) \in (0,1)$ such that if V is stationary* *in* $B_{R_0} \setminus B$, $(\omega_n R_0^n)^{-1}\|V\|(B_{R_0}) \le 1 + 1/64$, $\mathbf{C}^{(0)}$ *is a pair of half-planes meeting along* B *for which* $Q_V(\mathbf{C}^{(0)}) < \varepsilon/2$, V *has no triple junctions in* $B_1(0)$, *and* $\mathrm{sing}_B V \cap B_1(0) \cap B$ *has full* $\mathcal{H}^{n-1}$*-measure, then the following conclusions hold.*

(1) $\mathrm{sing} V \cap B_{1/2}(0) \setminus B = \emptyset$, *that is, there are no interior singularities, only boundary singularities.*

(2) There are smooth embedded n*-dimensional minimal submanifolds* M_i *for* $i = 1, 2$ *such that* $\partial M_i \cap B_{1/2}(0) = B \cap B_{1/2}(0)$ *for each* $i = 1, 2$.

(3) At every $Z \in \mathrm{sing}_B V \cap B_{1/2}$ *there exists a unique tangent cone* $\mathbf{C}_Z$, *which consists of two half-planes meeting along* B, *and satisfies the decay estimate*

$$\frac{1}{\rho^{n+2}} \int_{B_\rho(Z)} \mathrm{dist}^2(X, \mathrm{spt}\|\mathbf{C}_Z\|) \mathrm{d}\|V\| \le C\rho^{2\alpha} Q_V^2(\mathbf{C}^{(0)}),$$

for each $\rho \in (0, 1/4]$ *and where* $C = C(\mathbf{C}^{(0)})$.

Proof. Let $\hat{\mathbf{C}}^{(0)} \in \mathcal{C}$ denote the pair of intersecting planes containing $\mathbf{C}^{(0)}$. Furthermore, if we define $\vartheta \colon x \mapsto p_B(x) - p_{B^\perp}(x)$, where p_B is the orthogonal projection onto B, then Allard's reflection principle tells us that $\hat{V} = V + \vartheta_\# V$ is stationary in B_{R_0}. It is easy to verify that $\hat{V}$ taken with $\hat{\mathbf{C}}^{(0)}$ satisfies the assumptions of Theorem 3.6.1. Hence $\hat{V} \cap B_{1/2}$ consists of four smooth sheets meeting along B (since we assumed that the singular set was a dense subset of B), and so $\mathrm{spt}\|V\| \cap B_{1/2}$ is contained in these 4 sheets. However the constancy theorem (Theorem 2.2.14) implies that V must consist of exactly these sheets with constant multiplicities. Any such sheet must contribute at least a factor $1/2$ to the mass ratios in $B_{1/2}$, and so at most two of the sheets are contained in $\mathrm{spt}\|V\|$. Since $Q_V(\mathbf{C}^{(0)}) < \varepsilon$, choosing ε appropriately guarantees that the two sheets contained in $\mathrm{spt}\|V\|$ are indeed those closest to $\mathbf{C}^{(0)}$. Property (3) now follows directly from Theorem 3.6.1. □

3.7 Construction of the cover in Section 3.2

We prove Lemma 3.2.3 used in the proof of Lemma 3.2.6. We recall the statement for the reader's convenience.

Lemma. *Given* $c \le 1$, $\gamma < 1$ *it is possible to choose* $(\xi_i, \zeta_i) \in B_1(0) \setminus B$ *such that* $(T_{|\xi_i|, 2c/9}(\zeta_i))$ *are disjoint,* $(T_{|\xi_i|, c/2}(\zeta_i))$ *cover* $B_\gamma(0) \setminus B$ *and* $T_{|\xi_i|, c}(\zeta_i) \subset B_1(0) \setminus B$

for each i. Moreover there is $N = N(n)$ such that $(T_{|\xi_i|,c/2}(\zeta_i))$ can be divided into $N(n)$ disjoint subcollections.

Proof. Let (ξ_i, ζ_i) correspond to any maximal disjoint collection of $T_{|\xi_i|,2c/9}(\zeta_i)$ such that $T_{|\xi_i|,c/2}(\zeta_i) \cap B_\gamma(0) \neq \emptyset$. We claim that

$$B_\gamma(0) \setminus B \subset \bigcup_i T_{|\xi_i|,c/2}(\zeta_i).$$

We observe that by rotational symmetry, if we define $H := [0, \infty) \times \{0\}^k \times \mathbb{R}^{n-1}$, this is equivalent to the sets $D_{|\xi_i|,2c/9}(\zeta_i) := T_{|\xi_i|,2c/9}(\zeta_i) \cap H$ being disjoint, and $H \cap B_\gamma(0) \subset \bigcup_i D_{|\xi_i|,c/2}(\zeta_i)$. For simplicity then, we work in H and assume without loss of generality that $(\xi_i, \zeta_i) \in H$ for all i. Seeking a contradiction suppose that $(x, y) \in H \cap B_\gamma(0)$, and $(x, y) \notin D_{|\xi_i|,c/2}(\zeta_i)$ for every i, but that there exists a j such that $D_{|x|,2c/9}(y) \cap D_{|\xi_j|,2c/9}(\zeta_j) \neq \emptyset$, in particular there exists some $(a, b) \in D_{|x|,2c/9}(y) \cap D_{|\xi_j|,2c/9}(\zeta_j)$. Then, by the triangle inequality, and since x, ξ_j and a all lie on the same half line, it follows that

$$|(x, y) - (\xi_j, \zeta_j)| \leq \frac{c(1-\gamma)(|x| + |\xi_j|)}{9}. \tag{3.7.1}$$

Hence we find that

$$|x| \leq |\xi_j| + \frac{c(1-\gamma)(|x| + |\xi_j|)}{9} \leq \frac{(10-\gamma)|\xi_j|}{9} + \frac{(1-\gamma)|x|}{9},$$

where we used the fact that $c \leq 1$. Rearranging this yields

$$|x| \leq \frac{(10-\gamma)}{(8+\gamma)}|\xi_j| \leq \frac{5}{4}|\xi_j|. \tag{3.7.2}$$

Substituting (3.7.2) into (3.7.1) we see

$$|(x, y) - (\xi_j, \zeta_j)| \leq \frac{c(1-\gamma)|\xi_j|}{4},$$

implying that $(x, y) \in D_{|\xi_j|,c/2}(\zeta_j)$, a contradiction. Finally we wish to show that $D_{|\xi_i|,c}(\zeta_i) \subset H \cap B_\gamma(0)$ for each i. Fix some i. By hypothesis $D_{|\xi_i|,c/2}(\zeta_i) \cap B_\gamma(0) \neq \emptyset$ so there exists some $(x, y) \in B_\gamma(0) \cap H$ such that $(x, y) \in D_{|\xi_i|,c/2}(\zeta_i)$. It therefore follows that

$$|\xi_i| \le |(\xi_i, \zeta_i)| \le |(x,y)| + |(x,y) - (\xi_i, \zeta_i)| \le \gamma + \frac{(1-\gamma)|\xi_i|}{4},$$

where we used that $c \le 1$. Rearranging this implies that $|\xi_i| \le 4\gamma/(3+\gamma)$, and so we see that

$$|(\xi_i, \zeta_i)| \le \gamma + \frac{(1-\gamma)\gamma}{3+\gamma} = \frac{4\gamma}{3+\gamma}.$$

If we now pick any $(a,b) \in D_{|\xi_i|,c}(\zeta_i)$, then we can estimate

$$|(a,b)| \le |(\xi_i, \zeta_i)| + |(\xi_i, \zeta_i) - (a,b)| \le \frac{4\gamma}{3+\gamma} + \frac{(1-\gamma)|\xi_i|}{2} \le \frac{2\gamma(3-\gamma)}{3+\gamma}.$$

Requiring the right hand side to be less than 1 for all $\gamma \in (0,1)$ is equivalent to requiring that $(5\gamma - 2\gamma^2)/3$ is less than 1 for all such γ. This however is easily seen to be increasing, and equals 1 if $\gamma = 1$. Hence we conclude that $D_{|\xi_i|,c}(\zeta_i) \subset B_1(0) \cap H_1$ for every i.

We now claim that there is $N(n) \in \mathbb{N}$ such that for any fixed i, there are at most N values of j for which $D_{|\xi_i|,c/2}(\zeta_i) \cap D_{|\xi_j|,c/2}(\zeta_j) \neq \emptyset$. Fix i, then given such a j we must have

$$|\xi_j| \ge |\xi_i| - \Big||\xi_j| - |\xi_i|\Big| \ge |\xi_i| - \frac{c(1-\gamma)(|\xi_i| + |\xi_j|)}{4} \ge \frac{(3+\gamma)|\xi_i|}{4} - \frac{(1-\gamma)|\xi_j|}{4}.$$

This implies that

$$\frac{(5-\gamma)}{4}|\xi_j| \ge \frac{(3+\gamma)|\xi_i|}{4} \quad \text{and so} \quad |\xi_j| \ge \frac{(3+\gamma)|\xi_i|}{5-\gamma} \ge \frac{3|\xi_i|}{5} \ge \frac{|\xi_i|}{2}.$$

Therefore, if (ξ_j, ζ_j), (ξ_k, ζ_k) correspond to two disks, each of which intersects $D_{|\xi_i|,c/2}(\zeta_i)$, then, since $D_{|\xi_j|,2c/9}(\zeta_j)$ and $D_{|\xi_k|,2c/9}(\zeta_k)$ are disjoint we must have

$$|(\xi_j, \zeta_j) - (\xi_k, \zeta_k)| \ge \frac{c(1-\gamma)(|\xi_j| + |\xi_k|)}{9} \ge \frac{c(1-\gamma)|\xi_i|}{9}.$$

In other words, we have a lower bound on the distance between any two (ξ_j, ζ_j) and (ξ_k, ζ_k) corresponding to disks intersecting $D_{|\xi_i|,c/2}(\zeta_i)$. On the other hand, we find

$$|\xi_j| \le |\xi_i| + |\xi_i - \xi_j| \le |\xi_i| + \frac{c(1-\gamma)(|\xi_i| + |\xi_j|)}{4} \le \frac{(5-\gamma)|\xi_i|}{4} + \frac{(1-\gamma)|\xi_j|}{4},$$

which, upon rearranging yields

$$|\xi_j| \leq \frac{3|\xi_i|}{2}.$$

Therefore we have

$$|(\xi_i, \zeta_i) - (\xi_j, \zeta_j)| \leq \frac{c(1-\gamma)(|\xi_i| + |\xi_j|)}{4} \leq \frac{5c(1-\gamma)|\xi_i|}{8}.$$

Thus if J_i is the set of indices j for which $D_{|\xi_i|,c/2}(\zeta_i) \cap D_{|\xi_j|,c/2}(\zeta_j) \neq \emptyset$, then for any j, $k \in J_i$ we have

$$|(\xi_i, \zeta_i) - (\xi_j, \zeta_j)| \leq \frac{5c(1-\gamma)|\xi_i|}{8}, \qquad |(\xi_k, \zeta_k) - (\xi_j, \zeta_j)| \geq \frac{c(1-\gamma)|\xi_i|}{9}.$$

Rescaling by $(c(1-\gamma)|\xi_i|)^{-1}$, translating and identifying H with $\mathbb{R}^n$ this is equivalent to the following: a collection of points $x_j \in \mathbb{R}^n$ such that $x_j \in B_{5/8}(0)$ for each j and $|x_j - x_k| \geq 1/9$ for each $j \neq k$. Evidently there exists $N(n)$ such that $\#\{x_j\} \leq N(n)$. This of course implies that for any $x \in B_1(0) \setminus B$, there are at most $N(n)$ indices j for which $x \in T_{|\xi_j|,c/2}(\zeta_j)$.

Finally we note that this now easily implies that there is $N(n)$ such that the cover $(T_{|\xi_i|,c/2}(\zeta_i))$ can be split into $N(n)$ disjoint subcollections. Indeed any $T_{|\xi_i|,c/2}(\zeta_i)$ can overlap the regions $2^{-l-1} \leq |x| \leq 2^{-l}$ for at most two different values of $l =\geq 0$, and for each such region, only finitely many tori will intersect it. By applying the pigeonhole principle, for each l we can separate the tori intersecting the region $2^{-l-1} \leq |x| \leq 2^{-l}$ into at most $N(n)+1$ disjoint collections. Since intersecting tori can only overlap at most three such regions, we get that the cover $(T_{|\xi_i|,c/2}(\zeta_i)$ can be separated into at most $3(N(n)+1)$ disjoint subcollections.

□

Chapter 4

Mean curvature flow

In this chapter we will introduce the mean curvature flow, as well as some basic results concerning existence, behaviour and regularity of the flow. Of particular interest are the monotonicity formula of Huisken [27] and the local regularity theorem of White [60].

4.1 Mean curvature flow

Let M^n be an n-dimensional smooth manifold, and let $F_0 \colon M \to \mathbb{R}^{n+k}$ be a smooth immersion. A mean curvature flow is a one parameter family of immersions $F \colon M \times [0,T) \to \mathbb{R}^{n+k}$ satisfying the following partial differential equation

$$\begin{cases} \left(\frac{\partial F}{\partial t}(p,t)\right)^{\perp} = \vec{H}(p,t) & \forall\, (p,t) \in M \times (0,T) \\ F(p,0) = F_0(p) & \forall\, p \in M, \end{cases} \tag{4.1.1}$$

where $\vec{H}(p,t)$ denotes the mean curvature vector of $M_t := F(M,t)$ at the point $x(p,t) = F(p,t)$ and $(\cdot)^{\perp}$ denotes the projection to $(T_x M_t)^{\perp}$. In that which follows we will frequently use x to denote $F(p,t)$ and we suppress the arguments unless there is danger of ambiguity.

Remark 4.1.1. *If M has no boundary then it is possible to locally reparametrise F to eliminate any tangential components of motion (see [40, Proposition 1.3.4] for the details). If, for example, M is also compact then we can find a global reparametrisation that eliminates tangential components of motion. Indeed one can choose a one-parameter family $G(p,t)$ such that $F(M,t) = G(M,t)$ for each* *t, but such that*

$$\left(\frac{\partial G}{\partial t}(p,t)\right)^{\perp} = 0.$$

In particular we reformulate (4.1.1) *as*

$$\frac{\partial F}{\partial t}(p,t) = \vec{H}(p,t). \tag{4.1.2}$$

Moreover, (4.1.2) *may be written in the following, highly aesthetic form*

$$\frac{\partial F}{\partial t}(p,t) = \Delta_{M_t} F(p,t),$$

where Δ_{M_t} *denotes the Laplace-Beltrami operator of* M_t*, corresponding to the metric on* M_t *induced by the Euclidean metric on* $\mathbb{R}^{n+k}$*. In this way we see that we can consider the mean curvature flow to be a type of geometric heat equation. It is important to note however that this equation is in fact non-linear due to the metric dependence in* $\Delta_{g(t)}$*. This only introduces terms corresponding to first order spacial derivatives of* F*, so the equation is at least quasilinear, which makes it much more analytically tractable.*

4.1.1 Examples

We can get a good understanding of the basic behaviour of the flow by examining some examples. The first and most simple example is the flow of the round sphere. Consider $M = \mathbb{S}^n$ and let $F_0(p) := R_0 G(p)$ where G is the standard embedding of $\mathbb{S}^n$ into $\mathbb{R}^{n+k}$ and $R_0 > 0$. From the rotational symmetry of M and the rotational invariance of the equation, one might (correctly) suspect that the sphere remains round under the flow. Indeed if we suppose the existence of a solution of the mean curvature flow of the form $F(p,t) = R(t)G(p)$, then upon substitution into (4.1.1) we obtain an ordinary differential equation for $R(t)$ with the initial condition $R(0) = R_0$. This equation is easily solved to give

$$R(t) = \sqrt{R_0^2 - 2nt} \qquad t \in [0, R_0^2/2n).$$

As we can see, the mean curvature flow of the sphere shrinks to a point in finite time, beyond which there is no way to classically extend the flow. Finite time singularities like this are a feature of the flow, and more generally of reactiondiffusion equations to which the mean curvature flow is closely related. Since we can not in general expect the flow to exist for all time, understanding the nature of singularities of mean curvature flow is of crucial importance. One question in particular, which forms the basis of the problem studied in Chapter 5, is whether one can continue the flow in some meaningful way once such singularities develop.

Similar to the example of the shrinking sphere is that of shrinking cylinders. Indeed the cylinder $M = \mathbb{S}^k \times \mathbb{R}^{n-k}$ with initial radius R_0 remains cyclindrical, and shrinks about its axis with the radius evolving by the equation

$$R(t) = \sqrt{R_0^2 - 2kt}.$$

More generally let $F\colon M \times [0,T) \to \mathbb{R}^{n+k}$ be any flow defined on a maximal time interval that evolves by homothetic rescaling about a point. Indeed suppose that $F(p,t) = x_0 + \lambda(t)(F(p,0) - x_0)$ is a solution of mean curvature flow. Then it follows that $\lambda(t) = \sqrt{1 - t/T}$ and that at each time t, $\vec{H}(\cdot,t)$ satisfies the elliptic equation

$$\vec{H}(p,t) = \frac{(x_0 - x)^{\perp}}{2(T-t)}, \tag{4.1.3}$$

where T is the maximal existence time.

Definition 4.1.2 (Self-shrinkers)**.** *We call solutions of the mean curvature flow satisfying* (4.1.3) self-shrinking solutions*. Any such solution shrinks homothetically around x_0 to a point. If in particular M satisfies $\vec{H} = -x^{\perp}$, then $M_t := \sqrt{-2t}M$ defines a mean curvature flow for $t \in (-\infty, 0)$. In this case we call M a* self-shrinker*.*

In an entirely analogous manner, we can also consider self-expanding solutions.

Definition 4.1.3 (Self-expanders)**.** *We call any solution of the mean curvature flow satisfying the equation*

$$\vec{H}(p,t) = \frac{(x - x_0)^{\perp}}{2(t-T)} \tag{4.1.4}$$

a self-expanding solution*. Such a solution necessarily expands homothetically by scaling about the point x_0. If M satisfies $\vec{H} = x^{\perp}$, then $M_t := \sqrt{2t}M$ is a solution of the mean curvature flow for $t \in (0,\infty)$. In this case we call M a* self-expander*.*

Finally, given one solution of the mean curvature flow $F(p,t)$, we can construct a new solution by parabolically rescaling $(x,t) \mapsto (\lambda x, \lambda^2 t)$ for any $\lambda > 0$. Rescaling about singularities to examine the asymptotics of solutions at singular points is a fundamental technique in the regularity theory for mean curvature flow.

4.1.2 Short-time existence

Thus far we have only considered specific examples of the mean curvature flow, but under certain conditions on the initial condition M we can make more general statements about the existence and uniqueness of solutions. This is a direct consequence of the Nash-Moser implicit function theorem and work of Richard Hamilton [24, 25], first applied to the mean curvature flow by Gage-Hamilton [21], see also Smoczyk [56].

Proposition 4.1.4 (Short-time existence and uniqueness)**.** *Suppose that M is a closed (i.e. compact and without boundary) n-dimensional smooth manifold and that $F_0 \colon M \to \mathbb{R}^{n+k}$ is a smooth immersion. Then there is a unique smooth solution of* (4.1.1) *on a maximal time interval* $[0,T)$ *where* $T \in (0,\infty]$.

Remark 4.1.5. *In the codimension* 1 *case, i.e. where $k = 1$, Huisken-Polden [29] provided an alternative proof by writing the evolving hypersurfaces as normal graphs over the initial condition. They then obtain the result by applying standard theory of parabolic partial differential equations.*

One need not assume smoothness of the initial immersion F_0, it suffices to assume only that it is Lipschitz continuous. One can then show that the flow becomes instantaneously smooth, i.e. M_t is smooth for all times $t > 0$.

In case M is non-compact, the situation is more complicated. Indeed the short-time existence and uniqueness for non-compact, complete manifolds is still an open question. In the co-dimension 1 *case, work of Ecker-Huisken [16] shows that if M is an entire graph and satisfies a local Lipschitz condition initially, then the result holds.*

4.2 Monotonicity formula

A fundamental tool in the analysis of mean curvature flow is the Gaussian density. This serves as a parabolic analogue of the mass ratios and density that prove so successful in analysing the structure of stationary varifolds (see Chapter 2). In particular we will be able to develop analogues of both the monotonicity formula (Theorem 2.2.15), and Allard regularity (Theorem 2.3.2).

4.2.1 Gaussian density and local regularity

Definition 4.2.1. *We first define the* backwards heat kernel $\rho_{(x_0,t_0)}\colon \mathbb{R}^{n+k} \times (-\infty, t_0) \to (0, \infty)$ *as follows*

$$\rho_{(x_0,t_0)}(x,t) := \frac{1}{(4\pi(t_0 - t))^{n/2}} \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right).$$

Note that this differs slightly from the usual definition of the backwards heat kernel, in that the exponent in the scaling factor is $n/2$ rather than $(n+k)/2$. This is the correct scaling for integrating over n-dimensional surfaces and in particular integrating $\rho_{(x_0,t_0)}$ over an n-dimensional plane containing the point x_0 will return 1.

Definition 4.2.2. *For a mean curvature flow* $(M_t)_{0\leq t<T}$ *we define the* Gaussian density ratio at scale r centred at (x_0, t_0) *by*

$$\begin{aligned}\Theta(x_0, t_0, r) :&= \int_{M_{t_0-r^2}} \rho_{(x_0,t_0)}(x, t_0 - r^2) \mathrm{d}\mathcal{H}^n \\ &= \int_{M_{t_0-r^2}} \frac{1}{(4\pi r^2)^{n/2}} \exp\left(-\frac{|x - x_0|^2}{4r^2}\right) \mathrm{d}\mathcal{H}^n,\end{aligned}$$

for $0 < t_0 \leq T$, $0 < r \leq \sqrt{t_0}$ *and any* $x_0 \in \mathbb{R}^{n+k}$.

Huisken [27] proved the following monotonicity formula.

Theorem 4.2.3 (Monotonicity Formula). *If* $(M_t)_{0\leq t<t_0}$ *is a mean curvature flow, (defined on a not-necessarily maximal time-interval* $[0, t_0)$*) then*

$$\frac{d}{dt} \int_{M_t} \rho_{(x_0,t_0)}(x,t) \mathrm{d}\mathcal{H}^n(x) = -\int_{M_t} \left|\vec{H} - \frac{(x_0 - x)^\perp}{2(t_0 - t)}\right|^2 \rho_{(x_0,t_0)}(x,t) \mathrm{d}\mathcal{H}^n(x),$$

for each $t \in (0, t_0)$.

Remark 4.2.4. *Notice that the integrand on the right hand side is zero if and only if each* M_t *satisfies the shrinker equation* (4.1.3) *with* $T = t_0$*, and so the Gaussian density ratios are constant if and only if the flow is a self-shrinking solution.*

The monotonicity formula implies that $\Theta(x_0, t_0, r)$ is non-decreasing in r, which leads to the following definition.

Definition 4.2.5. *We define the* Gaussian density *to be*

$$\Theta(x_0, t_0) := \lim_{r \searrow 0} \Theta(x_0, t_0, r). \tag{4.2.1}$$

If (x_0, t_0) is a regular point of the flow, which is to say that in a space-time neighbourhood of (x_0, t_0) the flow may be smoothly parametrised, then it follows that $\Theta(x_0, t_0) = 1$. This follows because the surface M_{t_0} has a tangent plane at x_0, and at very small scales, the Gaussian density ratio centred at (x_0, t_0) at scale r is the same as the Gaussian density ratio centred at (x_0, t_0) at scale 1 of the flow rescaled parabolically by a factor $1/r$ about (x_0, t_0). If $1/r$ is very small, then in a large space-time neighbourhood of (x_0, t_0) the rescaled flow is very close to the tangent plane of M_{t_0} at x_0, and hence the density ratios will be close to 1. As $r \to 0$ it follows that the Gaussian density ratios will converge to 1. Conversely one can show that if the Gaussian density is 1, then at small scales the flow must be very close to a plane, from which it follows that there is a smooth local parametrisation of the flow, see [40] for the details.

Much like the case of stationary varifolds, we can actually formulate a more quantitative ε-regularity theorem. The following version is due to White [60]. There are others which we will mention later, but despite some fairly strong a priori assumptions on the regularity of the flow (i.e. that it is smooth), this particular version turns out to be surprisingly versatile and enough for our purposes in most situations.

Theorem 4.2.6 (Local regularity)**.** *Let* $\tau > 0$*. There are constants* $\varepsilon_0(n, k) > 0$ *and* $C_0(n, k, \tau) < \infty$ *such that if* $\partial M_t \cap B_{2r} = \emptyset$ *for* $t \in [0, r^2)$ *and*

$$\Theta(x, t, \rho) \leq 1 + \varepsilon_0 \qquad \rho \leq \tau\sqrt{t},\ x \in B_{2r}(x_0),\ t \in [0, r^2),$$

then

$$|A|(x,t) \le \frac{C_0}{\sqrt{t}} \qquad x \in M_t \cap B_r(x_0),\ t \in [0, r^2),$$

where $A(x,t)$ is the second fundamental form of M_t at the point x.

4.3 Lagrangian mean curvature flow

In Chapter 5 we will be specifically interested in Lagrangian mean curvature flow. Lagrangian submanifolds arise naturally in physics, in areas such as Hamiltonian mechanics, or more abstractly in geometry, such as the study of Calabi-Yau manifolds. In this section we give a quick overview of some of the relevant complex geometry that will be needed later.

We consider $\mathbb{C}^n$ with the standard complex coordinates $z_j = x_j + iy_j$. In what follows we will often identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. We let J denote the standard complex structure on $\mathbb{C}^n \cong T_z\mathbb{C}^n$, defined by

$$J\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} \qquad J\frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}.$$

We denote by ω the standard symplectic form on $\mathbb{C}^n$, defined by

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Definition 4.3.1. *We say that a smooth n-dimensional submanifold of $\mathbb{C}^n$ is* Lagrangian *if $\omega|_L = 0$.*

Definition 4.3.2. *Let the closed n-form Ω, called the* holomorphic volume form*, be defined by*

$$\Omega := dz_1 \wedge \cdots \wedge dz_n.$$

On any oriented Lagrangian a simple computation shows that $\Omega|_L = e^{i\theta_L}\mathrm{vol}_L$, where vol_L is the volume form on L.

Definition 4.3.3. *We call $e^{i\theta_L} \colon L \to S^1$ the* Lagrangian phase*, and θ_L the* Lagrangian angle*, which may be a multi-valued function. In the case that $\theta_L \colon L \to \mathbb{R}$ is a single valued function, we say that the Lagrangian L is* zero-Maslov*.*

We henceforth suppress the subscript L. An equivalent condition to θ being single valued is $[d\theta] = 0$, that is, $d\theta$ is cohomologous to 0.

Definition 4.3.4. *If $\theta \equiv \theta_0$ is constant, then we say that L is* special Lagrangian*.*

In this case L is calibrated by $\mathrm{Re}(e^{-i\theta_0}\mathrm{vol}_L)$, and hence is area-minimising in its homology class. We also consider the Liouville form λ on $\mathbb{C}^n$ defined by

$$\lambda := \sum_{j=1}^{n} x_j dy_j - y_j dx_j.$$

A simple calculation verifies that $d\lambda = 2\omega$.

Definition 4.3.5. *If there exists some function β on L such that $\lambda|_L = d\beta$ then we say that L is* exact*.*

In this thesis we will be more interested in local exactness, that is when the Liouville form λ only has a primitive in some open set. It can be shown that any smooth Lagrangian is locally exact.

The following remarkable property of smooth Lagrangians relates the Lagrangian angle and mean curvature vector (see for example [57])

$$\vec{H} = J\nabla\theta.$$

Consequently we see that the smooth minimal Lagrangians are exactly the smooth special Lagrangians.

Definition 4.3.6. *A* Lagrangian mean curvature flow *in $\mathbb{C}^n$ is a mean curvature flow $(L_t)_{0\leq t<T}$ with L_0 Lagrangian.*

Smoczyk [55] showed that the Lagrangian condition is preserved by the mean curvature flow, so for a Lagrangian mean curvature flow we have that L_t is Lagrangian for every t.

4.4 Brakke flows

The final ingredient needed for the next chapter is the notion of a Brakke flow. These are weak, measure theoretic notions of the mean curvature flow that enjoy good compactness properties. We start with the motivation for the definition.

4.4.1 Motivation

Suppose that M_t is a smooth mean curvature flow in $U \subset \mathbb{R}^{n+k}$ for $t \in [0,T)$. Then given any smooth test function $\varphi\colon U \times [0,T) \to \mathbb{R}$ such that the support of $\varphi(\cdot,t)$ is compactly contained in U for each t, we have by differentiating under the integral sign and using (4.1.1)

$$\frac{d}{dt}\int_{M_t} \varphi \mathrm{d}\mathcal{H}^n = \int_{M_t} -\varphi|\vec{H}|^2 + \nabla\varphi\cdot\vec{H} + \frac{\partial\varphi}{\partial t}\mathrm{d}\mathcal{H}^n.$$

Conversely, if $F : M \times [0,T) \to \mathbb{R}^{n+k}$ is a smooth one-parameter family of immersions and $M_t := F(M,t)$ satisfies

$$\frac{d}{dt}\int_{M_t} \varphi \mathrm{d}\mathcal{H}^n \leq \int_{M_t} -\varphi|\vec{H}|^2 + \nabla\varphi\cdot\vec{H} + \frac{\partial\varphi}{\partial t}\mathrm{d}\mathcal{H}^n.$$

for every smooth test function φ as above, then it follows that M_t must be a mean curvature flow. This forms the basis of our definition.

4.4.2 Definition

Though originally introduced by Brakke in [8], we use the slightly reformulated definition of Ilmanen [32].

Definition 4.4.1. *Let μ be a Radon measure on $\mathbb{R}^{n+k}$, and $\varphi \in C^2_c(\mathbb{R}^{n+k},[0,\infty))$. If any of the following 4 cases hold,*

1. *$\mu \llcorner \{\varphi > 0\}$ is not an n-rectifiable Radon measure,*

2. *$|\delta V|\llcorner\{\varphi > 0\}$ is not a Radon measure on $\{\varphi > 0\}$, where V is the rectifiable n-varifold associated with μ restricted to $\{\varphi > 0\}$,*

3. *$|\delta V|\llcorner\{\varphi > 0\}$ is singular with respect to $\mu\llcorner\{\varphi > 0\}$, or*

4. *$\int \varphi|\vec{H}|^2\mathrm{d}\mu = \infty$, where $\vec{H}$ is the generalised mean curvature vector of V, i.e. the Radon-Nikodym derivative of $|\delta V|$ with respect to μ;*

then we define $\mathcal{B}(\mu,\varphi) := -\infty$. If all four of the above cases fail, then we define

$$\mathcal{B}(\mu,\varphi) := \int -\varphi|\vec{H}|^2 + \nabla\varphi\cdot\vec{H}\mathrm{d}\mu.$$

We say that the family of Radon measures $\{\mu_t\}_{t\geq 0}$ *is a* Brakke flow *if for all* $t \geq 0$ *and* $\varphi \in C_c^2(\mathbb{R}^{n+k}, [0,\infty))$ *we have*

$$\overline{D}_t \mu_t(\varphi) \leq \mathcal{B}(\mu_t, \varphi),$$

where for a function $f\colon \mathbb{R} \to \mathbb{R}$, $\overline{D}_t$ *denotes the upper derivate of* f*, defined by*

$$\overline{D}_t f(t) := \limsup_{s\to t} \frac{f(s) - f(t)}{s - t}.$$

We say $\{\mu_t\}_{t\geq 0}$ *is an* integral Brakke flow *if* μ_t *corresponds to an integer rectifiable* n*-varifold for almost every* $t \geq 0$.

The primary reason for considering these weak flows is the following compactness theorem.

Theorem 4.4.2. *Let* $\{\mu_t^i\}_{t\geq 0}$ *be a sequence of integral Brakke flows, and suppose that for each* $U \subset\subset \mathbb{R}^{n+k}$ *there is a* $C = C(U) < \infty$ *with*

$$\sup_{i,t} \mu_t^i(U) \leq C.$$

Then there is a subsequence $\{\mu_t^{i_j}\}_{t\geq 0}$ *and an integral Brakke flow* $\{\mu_t\}$ *such that*

$$\mu_t^{i_j} \to \mu_t$$

as Radon measures for each $t \geq 0$*. Moreover, for almost every* $t \geq 0$*, there is a subsequence* $\{\mu_t^{i'_j}\}_{t\geq 0}$ *(where the subsequence depends on* t*) such that if* $V(\mu_t^{i'_j})$ *denotes the integer rectifiable* n*-varifold associated with* $\mu_t^{i'_j}$*, then*

$$V(\mu_t^{i'_j}) \to V(\mu_t)$$

as varifolds.

The proof can be found in [32].

Chapter 5

Short time existence of Lagrangian mean curvature flow

5.1 Motivation

A long standing open problem in the study of Calabi-Yau manifolds M is whether, given a Lagrangian submanifold $L \subset M$, we can find a special Lagrangian $\tilde{L}$ in the same homology or Hamiltonian isotopy class as L. As we saw in Section 4.3, special Lagrangians are precisely those Lagrangians that are area minimising in their homology class, because they are calibrated by the real part of the holomorphic volume form. Consequently this question may be naturally posed as a minimisation problem; that is, given a Calabi-Yau manifold M and a Lagrangian $L \subset M$, can we find a Lagrangian $\tilde{L}$ minimising area in the homology or Hamiltonian isotopy class of L? Such an $\tilde{L}$, if it exists, will automatically be special Lagrangian. It turns out that this minimisation problem is very subtle and fraught with difficulties. Indeed Schoen-Wolfson [49] showed that when the real dimension is 4, in any given class one can find a Lagrangian that minimises area among Lagrangians in that class, but that the minimiser need not be special Lagrangian. Later Wolfson [66] found a complex surface and a Lagrangian sphere in this surface such that the area minimiser among Lagrangians in the homology class of the sphere, is not special Lagrangian, and the area minimiser in the class is not Lagrangian. In light of examples like this it has been suggested that the mean curvature flow, being the gradient descent for area, could be used as an alternative way to construct special Lagrangian submanifolds.

In order to flow to a special Lagrangian, it is necessary for the flow to exist for all time, so that one may pass to the limit $t \to \infty$. Unfortunately, Neves [45] showed that one cannot expect long time existence in general. Indeed given any initial Lagrangian L, he showed that there is $\hat{L}$ in the same Hamiltonian isotopy class as L such that the mean curvature flow starting at $\hat{L}$ develops a finite time singularity. Since we can't hope to show that the flow exists classically for all time, we instead investigate whether it is possible to continue the flow in a weak sense. Specifically, is it possible to restart the flow from the singular Lagrangian that arises at the singular time?

The starting point is another result of Neves, who was able to classify singularities of zero-Maslov Lagrangian mean curvature flow [44]. Indeed it turns out that any singularity is asymptotic to a union of Lagrangian planes. Since special Lagrangians are necessarily zero-Maslov, and Lagrangian mean curvature flow preserves the Maslov class, considering singularities that arise under zero-Maslov mean curvature flow is not overly restrictive. Indeed if we are to flow to a special Lagrangian, the flow must be zero-Maslov itself. In fact we make a further simplification and study the simplest possible such singularity, namely one which is asymptotic to a transversely intersecting pair of planes. Motivated by this we prove the following theorem (which has appeared in [5]), which answers the existence part of a conjecture of Joyce [35, Problem 3.14]. Currently the corresponding uniqueness statement remains open.

Theorem 5.1.1. *Suppose that $L \subset \mathbb{C}^n$ is a compact Lagrangian submanifold of $\mathbb{C}^n$ with a finite number of singularities, each of which is asymptotic to a pair of transversely intersecting planes $P_1 + P_2$ such that neither $P_1 + P_2$ or $P_1 - P_2$ are area-minimising. Then there exists a $T > 0$ and a smooth Lagrangian mean curvature flow $(L_t)_{0<t<T}$ such that as $t \searrow 0$, $L_t \to L$ as varifolds, and in C^∞_{loc} away from the singularities.*

We remark that the assumptions $L \subset \mathbb{C}^n$ and L compact are made to simplify the analysis in the sequel, however since the analysis is all of an entirely local nature we may relax this to $L \subset M$ for some Calabi-Yau manifold M, and to L non-compact provided, in the latter case, that we impose suitable conditions at infinity.

The strategy of the proof is based heavily on work of Ilmanen-Neves-Schulze [31], who studied short time existence of the planar network flow. A network is a finite union of embedded line segments of non-zero length meeting only at their end-points. A regular network is one in which line-segments meet only in groups of three, making angles of $2\pi/3$ with one another. For regular networks, short time existence theory had already been established through work of Mantegazza-Novaga-Tortorelli [41], but the existence of short time solutions of non-regular networks remained open.

Self expanding solutions of the network flow asymptotic to arbitrary unions of half-lines meeting at the origin had been established by Mazzeo-Saez [42]. Since non-regular points are asymptotic to such unions, one would expect a solution with non-regular initial condition to be asymptotic to these self-expanding solutions at the non-regular points. This observation informs the approach of Ilmanen-Neves-Schulze. Indeed they 'regularise' a non-regular network by cutting out non-regular points, and replacing them with regular self-expanders at a scale s, which are asymptotic to the non-regular removed point. For each of these regular networks, the existence theory of Mantegazza-Novaga-Tortorelli applies, and we get a regular solution of the network flow existing for a short time. Moreover as the scale s goes to zero, the regularised initial conditions converge to the original non-regular network. Ilmanen-Neves-Schulze were able to establish uniform curvature estimates on this family of flows as well as a uniform lower bound on the existence time, allowing them to pass to a limit of flows to establish the existence of a regular flow that attains the non-regular initial condition in a suitable sense.

The approach taken here mirrors this exactly. We take a compact Lagrangian L with a singularity at the origin which is asymptotic to a pair of transversally intersecting planes $P = P_1 + P_2$. Work of Lotay-Neves [39] and Imagi-Joyce-Oliveira dos Santos [33] establishes the existence of a unique zero-Maslov Lagrangian self-expander Σ which is asymptotic to P. We cut out the singularity of L and glue in a piece of $\sqrt{2s}\Sigma$ to form a smooth Lagrangian L^s where $s > 0$ is small. Standard short time existence theory for smooth compact initial conditions implies the existence of smooth flows $(L^s_t)_{0 \leq t < T_s}$ with $T_s > 0$ and $L^s_0 = L^s$. We want to pass to the limit $s \searrow 0$, but since the maximum curvature of L^s scales like $s^{-1/2}$, the lower bound on T_s guaranteed by the short time existence theory scales like s, and hence $\inf_s T_s = 0$.

We therefore seek to establish a uniform lower bound on T_s along with uniform curvature estimates on (L^s_t) away from the singularity. We may then use the compactness theorem of Section 4.4 to pass to a limiting Brakke flow, which the curvature estimates then imply is in fact smooth. To do this we prove two key results. The first is a monotonicity formula for a geometric quantity that should be thought of as a primitive for the self-expander equation. This allows us to show that the glued-in sections of L_t^s evolve like the self-expander, remaining close in an L^2 sense on a large set of times. The second key component of the proof is a stability result for self-expanders, which says that if the solution remains close to the self-expander in L^2, then it is actually close in a stronger $C^{1,\alpha}$ sense. These two results combined allow us to show that for a uniform short time, the solutions L_t^s remain locally $C^{1,\alpha}$ close to the self-expander. This in turn implies uniform estimates on the Gaussian density ratios, which combined with Theorem 4.2.6 implies uniform curvature estimates near the origin. We also make use Ecker-Huisken style estimates for higher codimension flows which follow from work of Wang [58, 59] to control the curvature of the flows L_t^s away from the origin. These combined with the compactness theorem for Brakke flows are enough to establish Theorem 5.1.1.

The organisation of this section is as follows: In Section 5.2 we derive evolution equations for relevant geometric quantities and prove the aforementioned monotonicity formula. In section 5.3 we prove the stability result for self-expanders. In Section 5.4 we prove the main technical theorem, which establishes uniform Gaussian density ratio bounds near the origin. In Section 5.5 we prove Theorem 5.1.1. Section 5.6 contains the construction of the approximating initial conditions. Finally Section 5.7 contains miscellaneous technical results, including the high codimension Ecker-Huisken style curvature estimates.

5.2 Evolution equations

In this section we calculate evolution equations for various geometric quantities under the flow, including the Lagrange angle, primitives for the Liouville form, and the backwards heat kernel. From these evolution equations we can then establish the monotonicity formula that plays a crucial role in the proof of the main theorems.

Lemma 5.2.1. *Let $(L_t)_{0\leq t<T}$ be a Lagrangian mean curvature flow in $\mathbb{C}^n$. The following evolution equations hold.*

(i)

$$\frac{d\theta_t}{dt} = \Delta\theta_t,$$

where θ_t is the Lagrangian angle for L_t. Note that since only derivatives of θ_t appear here, this does not require the assumption that the flow is zero-Maslov.

(ii) In an open set where the flow is zero-Maslov and exact with β_t a primitive for the Liouville form,

$$\frac{d\beta_t}{dt} = \Delta\beta_t - 2\theta_t.$$

(iii)

$$\left(\frac{d\rho_{(x_0,t_0)}}{dt} + \Delta\rho_{(x_0,t_0)}\right) - |\vec{H}|^2\rho_{(x_0,t_0)} = -\left|\vec{H} - \frac{(x_0 - x)^\perp}{2(t_0 - t)}\right|^2 \rho_{(x_0,t_0)}.$$

Remark 5.2.2. *In particular, from part (iii) we have*

$$\left(\frac{d\rho_{(x_0,t_0)}}{dt} + \Delta\rho_{(x_0,t_0)}\right) - |\vec{H}|^2\rho_{(x_0,t_0)} \leq 0$$

Proof. (i) Differentiating the holomorphic volume form Ω and using Cartan's formula we have

$$\begin{aligned}\frac{d\Omega}{dt} = \mathcal{L}_{\vec{H}}\Omega = d(\vec{H} \lrcorner \Omega) &= d(ie^{i\theta_t}\nabla\theta_t \lrcorner \mathrm{vol}_{L_t}) \\ &= ie^{i\theta_t} d(\nabla\theta_t \lrcorner \mathrm{vol}_{L_t}) - e^{i\theta_t} d\theta_t \wedge (\nabla\theta_t \lrcorner \mathrm{vol}_{L_t}) \\ &= ie^{i\theta_t}\mathrm{div}(\nabla\theta_t)\mathrm{vol}_{L_t} - e^{i\theta_t} d\theta_t \wedge (\nabla\theta_t \lrcorner \mathrm{vol}_{L_t}),\end{aligned}$$

where $\lrcorner$ denotes interior multiplication. On the other hand

$$\frac{d\Omega}{dt} = \frac{d}{dt}\left(e^{i\theta_t}\mathrm{vol}_{L_t}\right) = ie^{i\theta_t}\frac{d\theta_t}{dt}\mathrm{vol}_{L_t} + e^{i\theta_t}\frac{d}{dt}\mathrm{vol}_{L_t}.$$

Comparing real and imaginary parts we have (i).

(ii) Using Cartan's formula again and denoting the Liouville form by λ_t, we have

$$\begin{aligned}d\left(\frac{d\beta_t}{dt}\right) = \mathcal{L}_{\vec{H}}\lambda_t &= d(\vec{H} \lrcorner \lambda_t) + \vec{H} \lrcorner d\lambda_t \\ &= d(\vec{H} \lrcorner \lambda_t) + J\nabla\theta_t \lrcorner 2\omega\end{aligned}$$

$$= d(\vec{H} \lrcorner \lambda_t) - 2d\theta_t.$$

Hence

$$d\left(\frac{d\beta_t}{dt} - \vec{H} \lrcorner \lambda_t + 2\theta_t\right) = 0.$$

By possibly adding a time-dependent constant to β_t this implies

$$\frac{d\beta_t}{dt} = \vec{H} \lrcorner \lambda_t - 2\theta_t.$$

Hence it only remains to show that $\vec{H} \lrcorner \lambda_t = \Delta\beta_t$. We first show that $\nabla\beta_t = (Jx)^T$. Indeed we have $d\beta_t = \lambda_t$, thus for a tangent vector τ

$$\langle \nabla\beta_t, \tau\rangle = d\beta_t(\tau) = \lambda_t(\tau) = \langle Jx, \tau\rangle = \langle (Jx)^T, \tau\rangle.$$

With this in hand we now choose normal coordinates at a point x, and denote the coordinate tangent vectors by $\{\partial_1, \ldots, \partial_n\}$. Then we calculate

$$\begin{aligned}
\nabla_i\nabla_j\beta_t = \langle \nabla_i(Jx)^T, \partial_j\rangle &= \partial_i\langle Jx, \partial_j\rangle - \langle (Jx)^T, D_{\partial_i}\partial_j\rangle \\
&= \langle J\partial_i, \partial_j\rangle + \langle Jx, D_{\partial_i}\partial_j\rangle - \langle (Jx)^T, D_{\partial_i}\partial_j\rangle \\
&= \omega(\partial_i, \partial_j) + \langle (Jx)^\perp, D_{\partial_i}\partial_j\rangle \\
&= \langle Jx, h_{ij}\rangle,
\end{aligned}$$

where h_{ij} is the second fundamental form. Taking the trace of each side we have

$$\Delta\beta_t = \langle Jx, \vec{H}\rangle = \vec{H} \lrcorner \lambda_t.$$

(iii) We may assume without loss of generality that $x_0 = 0$ and $t_0 = 0$, and we will suppress the subscripts of ρ. We first calculate

$$\frac{\partial\rho}{\partial t} = \left(-\frac{n}{2t} - \frac{|x|^2}{4t^2}\right)\rho \qquad \frac{\partial\rho}{\partial x^i} = \frac{x^i}{2t}\rho \qquad \frac{\partial^2\rho}{\partial x^i\partial x^j} = \left(\frac{\delta_{ij}}{2t} + \frac{x^i x^j}{4t^2}\right)\rho.$$

Then we have

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(D\rho) = \left(-\frac{n}{2t} - \frac{|x|^2}{4t^2}\right)\rho + \sum_{i,j=1}^{n+k} p^{ij}\frac{\partial^2\rho}{\partial x^i\partial x^j}$$

$$= \left(-\frac{n}{2t} - \frac{|x|^2}{4t^2}\right)\rho + \sum_{i,j=1}^{n+k} p^{ij}\left(\frac{\delta_{ij}}{2t} + \frac{x^i x^j}{4t^2}\right)\rho$$
$$= \left(-\frac{n}{2t} - \frac{|x|^2}{4t^2}\right)\rho + \frac{n}{2t}\rho + \frac{|x^T|^2}{4t^2}\rho = -\frac{|x^\perp|^2}{4t^2},$$

where p^{ij} denotes the matrix of the projection onto T_xM. We therefore calculate

$$\left(\frac{d}{dt} + \Delta\right)\rho = \frac{\partial\rho}{\partial t} + \langle D\rho, \vec{H}\rangle + \operatorname{div}(\nabla\rho)$$
$$= \frac{\partial\rho}{\partial t} + \operatorname{div}(D\rho) + 2\langle D\rho, \vec{H}\rangle$$
$$= \frac{\partial\rho}{\partial t} + \operatorname{div}(D\rho) - \left|\vec{H} - \frac{x^\perp}{2t}\right|^2\rho + \frac{|x^\perp|^2}{4t^2}\rho + |\vec{H}|^2\rho$$
$$= -\left|\vec{H} - \frac{x^\perp}{2t}\right|^2\rho + |\vec{H}|^2\rho,$$

which establishes the claim. □

Remark 5.2.3. *From the above evolution equations we see that both the zero-Maslov condition, and local exactness are preserved by the flow. Indeed that the zero-Maslov condition is preserved follows easily, and for local exactness we observe*

$$\frac{d\lambda_t}{dt} = \mathcal{L}_{\vec{H}}\lambda_t = d(\vec{H}\lrcorner\lambda_t) + \vec{H}\lrcorner d\lambda_T$$
$$= d(\vec{H}\lrcorner\lambda_t) + J\nabla\theta_t\lrcorner 2\omega$$
$$= d(\vec{H}\lrcorner\lambda_t) - 2d\theta_t.$$

So by the fundamental theorem of calculus we have

$$\lambda_t = \lambda_0 + \int_0^t \frac{d\lambda_s}{ds}\mathrm{d}s,$$

where the right hand side is exact if λ_0 is.

Let ϕ be a cut-off function supported on B_3 with $0 \le \phi \le 1$, $\phi \equiv 1$ on B_2 and the estimates $|D\phi| \le 2$ and $|D^2\phi| \le C$. We then have the following lemma.

Lemma 5.2.4. *Suppose that (L_t) are exact in B_3 and define $\alpha_t := \beta_t + 2t\theta_t$.*

Then

$$\frac{d}{dt}\int_{L_t}\phi\alpha_t^2\rho\mathrm{d}\mu \leq -\int_{L_t}\phi|2t\vec{H}-x^\perp|^2\rho\mathrm{d}\mu + C\int_{L_t\cap(B_3\setminus B_2)}\alpha_t^2\rho\mathrm{d}\mu.$$

where $C = C(\phi)$*.*

Remark 5.2.5. *Note that it follows from Lemma 5.2.1 that*

$$\frac{d}{dt}\alpha_t = \Delta\beta_t - 2\theta_t + 2\theta_t + 2t\Delta\theta_t = \Delta\alpha_t.$$

This is the motivation for why we might expect α_t *to satisfy some sort of monotonicity formula in the first place.*

Proof. We calculate

$$\left(\frac{d}{dt}-\Delta\right)\phi = \frac{\partial\phi}{\partial t} - \mathrm{div}D\phi = -\Delta_{\mathbb{R}^{2n}}\phi + \mathrm{tr}_{(TL)^\perp}D^2\phi \leq C\mathbb{1}_{B_3\setminus B_2},$$

where $\mathbb{1}_{B_3\setminus B_2}$ denotes the indicator function on $B_3\setminus B_2$. Then

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)(\phi\alpha_t^2) &= \phi\left(\frac{d}{dt}-\Delta\right)\alpha_t^2 + \alpha_t^2\left(\frac{d}{dt}-\Delta\right)\phi - 2\langle\nabla\phi,\nabla\alpha_t^2\rangle\\ &\leq 2\phi\alpha_t\left(\frac{d}{dt}-\Delta\right)\alpha_t - 2\phi|\nabla\alpha_t|^2 + C\alpha_t^2\mathbb{1}_{B_3\setminus B_2} - 4\alpha_t\langle\nabla\phi,\nabla\alpha_t\rangle.\end{aligned}$$

Using Young's inequality we estimate the last term on the set $\{\phi>0\}$ as follows

$$-4\alpha_t\langle\nabla\phi,\nabla\alpha_t\rangle \leq 4|D\phi||\alpha_t||\nabla\alpha_t| \leq \phi|\nabla\alpha_t|^2 + \frac{4|D\phi|^2}{\phi}\alpha_t^2 \leq \phi|\nabla\alpha_t|^2 + C\alpha_t^2\mathbb{1}_{B_3\setminus B_2},$$

where we used that

$$\frac{|D\phi|^2}{\phi} \leq 2\max|D^2\phi| \leq C.$$

This is true of any compactly supported smooth (or even C^2) function (see [32, Lemma 6.6] for a proof). Thus we arrive at

$$\left(\frac{d}{dt}-\Delta\right)\phi\alpha_t^2 \leq -\phi|\nabla\alpha_t|^2 + C\alpha_t^2\mathbb{1}_{B_3\setminus B_2}.$$

We now just differentiate under the integral and use Green's identity to get

$$\begin{aligned}\frac{d}{dt}\int_{L_t}\phi\alpha_t^2\rho \mathrm{d}\mu &= \int_{L_t}\rho\frac{d(\phi\alpha_t^2)}{dt}+\phi\alpha_t^2\frac{d\rho}{dt}-|\vec{H}|^2\phi\alpha_t^2\rho \mathrm{d}\mu\\ &= \int_{L_t}\phi\alpha_t^2\Delta\rho-\rho\Delta(\phi\alpha_t^2)\mathrm{d}\mu+\int_{L_t}\rho\frac{d(\phi\alpha_t^2)}{dt}+\phi\alpha_t^2\frac{d\rho}{dt}-|\vec{H}|^2\phi\alpha_t^2\rho \mathrm{d}\mu\\ &= \int_{L_t}\rho\left(\frac{d}{dt}-\Delta\right)(\phi\alpha_t^2)+\left(\left(\frac{d}{dt}+\Delta\right)\rho-|\vec{H}|^2\rho\right)\phi\alpha_t^2\mathrm{d}\mu\\ &\leq -\int_{L_t}\phi\rho|\nabla\alpha_t|^2\mathrm{d}\mu+C\int_{L_t\cap(B_3\setminus B_2)}\alpha_t^2\rho \mathrm{d}\mu.\end{aligned}$$

Since $\nabla\alpha_t=\nabla\beta_t+2t\nabla\theta_t=Jx^\perp-2tJ\vec{H}$ we are left with precisely the desired inequality . □

5.3 Stability of self-expanders

In this section we prove a dynamic stability result for Lagrangian self-expanders. More specifically we show that if a Lagrangian submanifold is asymptotic to some pair of planes and is almost a self-expander in a weak sense, then the submanifold is actually close in a stronger topology to some self-expander. Let P_1, $P_2\subset\mathbb{C}^n$ be Lagrangian planes intersecting transversally such that neither P_1+P_2 nor P_1-P_2 are area minimising. We denote by $P:=P_1+P_2$. We will need the following uniqueness result, proved by Lotay-Neves [39] in dimension 2 and Imagi-Joyce-Oliveira dos Santos [33] in dimensions 3 and higher.

Theorem 5.3.1. *There exists a unique smooth, zero-Maslov class Lagrangian self-expander asymptotic to P.*

The stability theorem is proved by a compactness argument, and relies on Theorem 5.3.1 to get a contradiction. This is the main missing ingredient in generalising the work of this chapter to singularities asymptotic to other combinations of intersecting planes. Much of the analysis does not rely specifically on the fact that the singularity is asymptotic to two transversally intersecting planes. If Theorem 5.3.1 could be generalised to other combinations of planes there is hope that a corresponding short time existence result could be proved. In fact, it might not be necessary to have the full power of a uniqueness statement, rather some sort of isolatedness theorem should suffice. For now however, whether such theorems can be proved remain challenging open questions.

Before stating the theorem, we introduce what it means for two manifolds to be ε-close in $C^{1,\alpha}$.

Definition 5.3.2. *Given an open set $U \subset \mathbb{R}^{n+k}$ and two n-dimensional submanifolds Σ and L defined in U, we say that Σ and L are* 1-close in $C^{1,\alpha}(W)$ *for any $W \subset U$ with $\operatorname{dist}(W, \partial U) \geq 1$ if for all $x \in W$, $B_1(x) \cap \Sigma$ and $B_1(x) \cap L$ are both graphical over some common n-dimensional plane, and if u and v denote the respective graph functions then $\|u - v\|_{1,\alpha} \leq 1$. We then say that Σ and L are* ε-close in $C^{1,\alpha}(W)$ *if $\varepsilon^{-1}\Sigma$ and $\varepsilon^{-1}L$ are 1-close in $\varepsilon^{-1}W$ for any W with* $\operatorname{dist}(\varepsilon^{-1}W, \varepsilon^{-1}\partial U) \geq 1$.

Theorem 5.3.3 (Stability theorem)**.** *Fix R, r, $\tau > 0$, α, $\varepsilon_0 < 1$, and $C, M < \infty$. Let Σ be the unique smooth zero-Maslov Lagrangian self-expander asymptotic to P. Then for all $\varepsilon > 0$ there exists $\tilde{R} \geq R$, η, $\nu > 0$ each dependent on ε_0, ε, r, R, τ, α, C, M and P such that if L is a smooth Lagrangian submanifold which is zero-Maslov in $B_{\tilde{R}}$ and*

(i) $|A| \leq M$ on $L \cap B_{\tilde{R}}$,

(ii) For all x and $0 < r \leq \tau$

$$\int_L \rho_{(x,0)}(y, -r^2) \mathrm{d}\mathcal{H}^n \leq 1 + \varepsilon_0$$

(iii) L satisfies

$$\int_{L \cap B_{\tilde{R}}} |\vec{H} - x^\perp|^2 \mathrm{d}\mathcal{H}^n \leq \eta, \tag{5.3.1}$$

(iv) The connected components of $L \cap A(r, \tilde{R})$ (where $A(r, \tilde{R}) := \overline{B_{\tilde{R}}} \setminus B_r$) are in one to one correspondence with the connected components of $P \cap A(r, \tilde{R})$ and

$$\operatorname{dist}(x, P) \leq \nu + C \exp\left(\frac{-|x|^2}{C}\right),$$

for all $x \in L \cap A(r, \tilde{R})$;

then L is ε-close to Σ in $C^{1,\alpha}(\overline{B}_{\tilde{R}})$.

Proof. Seeking a contradiction, suppose that the result were not true. Then there would exist sequences $\nu_i \searrow 0$, $\eta_i \searrow 0$, $R_i \to \infty$ and L_i such that each L_i is a smooth Lagrangian submanifold of $\mathbb{C}^n$ that is zero-Maslov in B_{R_i}, satisfying

(1) $|A^{L_i}| \leq M$ on $L_i \cap B_{R_i}$,

(2) For all x and $0 < r \leq \tau$,

$$\int_{L_i} \rho_{(x,0)}(y, -r^2) \mathrm{d}\mathcal{H}^n \leq 1 + \varepsilon_0$$

(3) L_i satisfies

$$\int_{L_i \cap B_{R_i}} |\vec{H} - x^\perp|^2 \mathrm{d}\mathcal{H}^n \leq \eta_i$$

(4) The connected components of $L_i \cap A(r, R_i)$ are in one to one correspondence with the connected components of $P \cap A(r, R_i)$ and

$$\mathrm{dist}(x, P) \leq \nu_i + C \exp\left(\frac{-|x|^2}{C}\right)$$

for all $x \in L_i \cap A(r, R_i)$,

(5) L_i is not ε-close to Σ in $C^{1,\alpha}(B_{R_i})$.

By virtue of (1), (4), and a suitable interpolation inequality, it follows that for some $\rho > 0$, outside of B_ρ, L_i and Σ are both $\varepsilon/4$-close to P in $C^{1,\alpha}$. Hence, in order that (5) is satisfied, we conclude that for large i, L_i is not ε-close to Σ in $C^{1,\alpha}(B_\rho)$.

On the other hand, by (1) and (2) we may extract a subsequence of L_i that converges in $C^{1,\alpha}_{loc}$ for all $\alpha < 1$ to some limit L_∞, a $C^{1,1}$ zero-Maslov Lagrangian submanifold. The estimate (2) passes to the limit and tells us that L_∞ has unit multiplicity everywhere, and bounded area ratios. Since L_∞ is $C^{1,1}$ we can define mean curvature in a weak sense, and (3) implies

$$\int_{L_\infty} |\vec{H} - x^\perp|^2 \mathrm{d}\mathcal{H}^n = 0.$$

By standard Schauder theory for elliptic PDE, this immediately implies that L_∞ is in fact smooth and satisfies the expander equation in the classical sense. Consequently L_∞ is a smooth, zero-Maslov class Lagrangian submanifold, and (4) implies that L_∞ is asymptotic to P. Theorem 5.3.1 then implies that $L_\infty = \Sigma$, which contradicts (5). □

5.4 Uniform Gaussian density ratio bounds

Suppose, as in the previous section, that $P := P_1 + P_2$ is a pair of transversely intersecting Lagrangian planes such that neither $P_1 + P_2$ nor $P_1 - P_2$ are minimising, and that Σ is a zero-Maslov Lagrangian self-expander asymptotic to P. For the purposes of this section we assume the existence of an approximating family $(L^s)_{0<s\leq c}$ of compact Lagrangians, each exact and zero-Maslov in B_4 satisfying the following properties. The existence of such a family will be established in section 5.6.

(H1) The area ratios are uniformly bounded, i.e. there exists a constant D_1 such that

$$\mathcal{H}^n(L^s \cap B_r(x)) \leq D_1 r^n \qquad \text{for all } r > 0,\ s \in (0, c], \text{ and for all } x.$$

(H2) There is a constant D_2 such that for every s and $x \in L^s \cap B_4$

$$|\theta^s(x)| + |\beta^s(x)| \leq D_2(|x|^2 + 1),$$

where θ^s and β^s are, respectively, the Lagrangian angle of L^s and a primitive for the Liouville form on L^s.

(H3) For any $\alpha \in (0, 1)$, the rescaled manifolds $\tilde{L}^s := (2s)^{-1/2} L^s$ converge in $C^{1,\alpha}_{loc}$ to Σ. Moreover the second fundamental form of $\tilde{L}^s$ is bounded uniformly in s and without loss of generality we can assume that

$$\lim_{s\to 0}(\tilde{\theta}^s + \tilde{\beta}^s) = 0$$

locally on $\tilde{L}^s$. (Note that $\tilde{L}^s$ is exact in the ball $B_{4(2s)^{-1/2}}$ so we can make sense of $\tilde{\beta}^s$ in the limit.)

(H4) The connected components of $P \cap A(r_0\sqrt{s}, 4)$ are in one to one correspondence with the connected components of $L^s \cap A(r_0\sqrt{s}, 4)$, and each component can be parametrised as a graph over the corresponding plane P_i

$$L^s \cap A(r_0\sqrt{s}, 3) \subset \{x + u_s(x) \mid x \in P \cap A(r_0\sqrt{s}, 3)\} \subset L^s \cap A(r_0\sqrt{s}, 4),$$

where the function $u_s \colon P \cap A(r_0\sqrt{s}, 3) \to P^\perp$ is normal to P and satisfies the estimate

$$|u_s(x)| + |x| \left|\overline{\nabla} u_s(x)\right| + |x|^2 |\overline{\nabla}^2 u_s(x)| \leq D_3 \left(|x|^2 + \sqrt{2s} e^{-b|x|^2/2s} \right),$$

where $\overline{\nabla}$ denotes the covariant derivative on P, and $b > 0$.

We will denote by $(L_t^s)_{t\in[0,T_s)}$ a smooth solution of Lagrangian mean curvature flow with initial condition L^s. For $x_0 \in \mathbb{R}^{2n}$ and $t > 0$ we define

$$\Phi(x_0, t)(x) := \rho_{(x_0,0)}(x, -t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - x_0|^2}{4t} \right)$$

We introduce a slightly modified notion of the Gaussian density ratios, which we will continue to refer to as the Gaussian density ratios, of L_t^s at x_0, denoted $\Theta_t^s(x_0, r)$ and defined as

$$\Theta_t^s(x_0, r) := \int_{L_t^s} \Phi(x_0, r^2) \mathrm{d}\mathcal{H}^n = \int_{L_t^s} \frac{1}{(4\pi r^2)^{n/2}} e^{-|x-x_0|^2/4r^2} \mathrm{d}\mathcal{H}^n(x), \qquad (5.4.1)$$

defined for $t < T_s$. The monotonicity formula of Huisken tells us that

$$\Theta_t^s(x_0, r) = \Theta^s(x_0, t + r^2, r) \leq \Theta^s(x_0, t + r^2, \rho) = \int_{L^s_{t+r^2-\rho^2}} \Phi(x_0, t + r^2) \mathrm{d}\mathcal{H}^n,$$

for all $\rho \geq r$. In particular choosing $\rho^2 = t + r^2$ we have

$$\Theta_t^s(x_0, r) \leq \int_{L^s} \Phi(x_0, t + r^2) \mathrm{d}\mathcal{H}^n.$$

We also define

$$\tilde{L}_t^s = \frac{L_t^s}{\sqrt{2(s + t)}}.$$

We will denote by $\tilde{\Theta}_t^s(x_0, r)$ the Gaussian density ratios of $(\tilde{L}_t^s)$, that is

$$\tilde{\Theta}_t^s(x_0, r) := \int_{\tilde{L}_t^s} \Phi(x_0, r) \mathrm{d}\mathcal{H}^n.$$

One of the primary reasons for modifying the Gaussian density ratios is that our new ratios behave well under the above rescaling. Indeed we can calculate

$$\Theta_t^s(x_0, r) = \tilde{\Theta}_t^s \left(\frac{x_0}{\sqrt{2(s+t)}}, \frac{r}{\sqrt{2(s+t)}} \right).$$

The primary goal of this section is now to prove uniform bounds on the Gaussian density ratios of L_t^s, which we formulate as the following result.

Theorem 5.4.1. *Let $\varepsilon_0 > 0$. There are s_0, δ_0 and τ depending on D_1, D_2, D_3, Σ and r_0 such that if*

$$t \leq \delta_0,\ r^2 \leq \tau t \ \text{ and } \ s \leq s_0,$$

then

$$\Theta_t^s(x_0, r) \leq 1 + \varepsilon_0$$

for every $x_0 \in B_1$.

We start by proving estimates like the one in the above theorem hold for a short time or far from the origin. Geometrically speaking this should be expected. Away from the origin the L^s either coincide with the original Lagrangian L, or we are in the graphical region of hypothesis (H4) where the L^s are close to planar. On the other hand, globally the maximum curvature of L^s, provided s is sufficiently small, is proportional to $s^{-1/2}$, so one expects control of the Gaussian density ratios up to a scale proportional to s.

Lemma 5.4.2 (Far from the origin estimate)**.** *Let $\varepsilon_0 > 0$. There are $\delta_1 > 0$, $K_0 < \infty$ such that if $r^2 \leq t \leq \delta_1$ and $s > 0$, then*

$$\Theta_t^s(x_0, r) \leq 1 + \varepsilon_0,$$

for all $x_0 \in A(K_0\sqrt{2t}, 1)$.

Proof. We first claim that there is a $K_0 < \infty$ such that if $y_0 \in \mathbb{R}^{2n}$ has $|y_0| \geq K_0$, then for any $\lambda > 0$ and s we have

$$\int_{\lambda(L^s \cap B_3(0))} \Phi(y_0, 1) \mathrm{d}\mathcal{H}^n \leq 1 + \varepsilon_0/2.$$

Indeed if this were not the case, then there would exist sequences y_i, λ_i and s_i

with $|y_i| \to \infty$ such that

$$\int_{\lambda_i(L^{s_i} \cap B_3(0))} \Phi(y_i, 1) \mathrm{d}\mathcal{H}^n \geq 1 + \varepsilon_0/2. \tag{5.4.2}$$

First we note that λ_i must be unbounded since, for some universal constant C we have

$$\begin{aligned}
\int_{\lambda_i(L^{s_i} \cap B_3(0))} \Phi(y_i, 1) \mathrm{d}\mathcal{H}^n &\leq \int_{\lambda_i(L^{s_i} \cap B_3(0))} \frac{1}{(4\pi)^{n/2}} e^{-|y_i|^2/8} e^{3|x|^2/4} \mathrm{d}\mathcal{H}^n \\
&\leq e^{-|y_i|^2/8} \lambda_i^n \int_{L^{s_i} \cap B_3} \frac{1}{(4\pi)^{n/2}} e^{9\lambda_i^2/4} \mathrm{d}\mathcal{H}^n \\
&\leq C \lambda_i^n e^{-|y_i|^2/8 + c\lambda_i^2} \underbrace{\mathcal{H}^n(L^{s_i} \cap B_3(0))}_{\leq D_1 3^n},
\end{aligned}$$

so it is easily seen that if λ_i were bounded then (5.4.2) would fail for large i. Next from the estimate (H4) we have that

$$|\overline{\nabla}^2 u_s^j(x)| \leq C \left(1 + \frac{\sqrt{2s}}{|x|^2} e^{-b|x|^2/2s}\right),$$

for every $x \in A(r_0\sqrt{2s}, 4)$ and hence

$$|A| \leq C \left(1 + \frac{1}{\sqrt{2s}} e^{-b|x|^2/2s}\right)$$

on $B_3 \cap L^s$, since on $B_{r_0\sqrt{2s}}$ we have $|A| \leq C(2s)^{-1/2}$ where C is a curvature bound for Σ. We rescale and define

$$\hat{L}_i := \lambda_i L^{s_i} \qquad \sigma_i := \lambda_i^2 s_i,$$

so that on $\hat{L}_i$ we have the estimate

$$|A| \leq \frac{C}{\lambda_i} \left(1 + \frac{1}{\sqrt{s_i}} e^{-b|x|^2/2\lambda_i^2 s_i}\right) = C \left(\lambda_i^{-1} + \sigma_i^{-1/2} e^{-b|x|^2/2\sigma_i}\right).$$

Consequently $|A| \to 0$ uniformly on compact sets centred at y_i, so it follows that locally $\hat{L}_i - y_i$ converges to a plane, but this contradicts (5.4.2).

We next observe that (H1) ensures that we may choose $\delta_1 > 0$ small enough such that for any $x_0 \in B_1(0)$ and $l \leq 2\sqrt{\delta_1}$ we have

$$\int_{L^s \setminus B_3} \Phi(x_0, l) \mathrm{d}\mathcal{H}^n \leq \varepsilon_0/2$$

By the monotonicity formula we have that for any $r^2, t \leq \delta_1$

$$\begin{aligned}
\Theta_t^s(x_0, r) &\leq \int_{L^s} \Phi(x_0, r^2 + t) \mathrm{d}\mathcal{H}^n \\
&= \int_{L^s \setminus B_3} \Phi(x_0, r^2 + t) \mathrm{d}\mathcal{H}^n + \int_{L^s \cap B_3} \Phi(x_0, r^2 + t) \mathrm{d}\mathcal{H}^n \\
&\leq \varepsilon_0/2 + \int_{(r^2+t)^{-1}(L^s \cap B_3)} \Phi\left(\frac{x_0}{\sqrt{r^2 + t}}, 1\right) \mathrm{d}\mathcal{H}^n \\
&\leq 1 + \varepsilon_0
\end{aligned}$$

provided that $|x_0| \geq K_0\sqrt{r^2 + t}$, so imposing the additional requirement that $r^2 \leq t$ this gives precisely the desired result. $\square$

Remark 5.4.3. *We observe that increasing K_0 will only strengthen the hypotheses, and so we may do so freely if necessary without changing the conclusions. This will be important in the next lemma, and also in the proof of the main theorem where we will assume that K_0 is at least* 1.

Lemma 5.4.4 (Short-time estimate)**.** *Let $\varepsilon_0 > 0$. There are $s_1 > 0$ and $q_1 \in (0, 1)$ such that if $s \leq s_1$, $r^2 \leq q_1 s$ and $t \leq q_1 s$ then*

$$\Theta_t^s(x, r) \leq 1 + \varepsilon_0, \tag{5.4.3}$$

for all $x \in B_1$.

Proof. Fix $\alpha \in (0, 1)$ and let $q_1 = q_1(\Sigma, \varepsilon_0, \alpha)$ be as in Lemma 5.7.2. We may assume without loss of generality that $q_1 < 1$. By Lemma 5.4.2 we need only prove the estimate for $x \in B_{K_0\sqrt{2t}}$. We fix $\alpha \in (0, 1)$ and seek to apply 5.7.2 with $R = K_0\sqrt{q_1} + 1$, which we can assume is at least 2 by increasing K_0, and the rescaled flow $\hat{L}_t := (2s)^{-1/2} L_{2st}^s$. This is a mean curvature flow with initial condition $\tilde{L}^s$. By (H3) we know that $\tilde{L}^s \to \Sigma$ in $C_{loc}^{1,\alpha}$. In particular, letting $\varepsilon = \varepsilon(\varepsilon_0, \Sigma, \alpha)$ from Lemma 5.7.2, if s is small enough we can ensure that $\tilde{L}^s$ is ε-close to Σ in $C^{1,\alpha}(B_R(0))$. The conclusion of Lemma 5.7.2 then says that for

$r^2, t \leq q_1$ and $x \in B_{K_0\sqrt{q_1}}$ we have

$$\hat{\Theta}^s_t(x,r) = \int_{\hat{L}^s_t} \Phi(x,r^2)\mathrm{d}\mathcal{H}^n = \int_{L^s_{2st}} \Phi(2sx, 2sr^2)\mathrm{d}\mathcal{H}^n \leq 1 + \varepsilon_0,$$

or in other words

$$\Theta^s_t(x,r) \leq 1 + \varepsilon_0,$$

for all $r^2, t \leq q_1 s$ and $x \in B_{K_0\sqrt{2sq_1}}$. However since $t \leq q_1 s$ this holds for all $x \in B_{K_0\sqrt{2t}}$. □

The next lemma shows that in an annular region, and for short times, we retain uniform control on both the distance to P and the Gaussian density ratios. This follows primarily from (H4), since L^s in this region is graphical with good estimates, and hence well behaved.

Lemma 5.4.5 (Proximity to $P = P_1 + P_2$)**.** *There are constants C_1, and r_1 such that for any $\nu > 0$ there are $s_2, \delta_2 > 0$ such that the following holds. If $s \leq s_2$ and $t \leq \delta_2$ then we have the estimate*

$$\mathrm{dist}(y_0, P) \leq \nu + C_1 e^{-|y_0|^2/C_1} \qquad \forall y_0 \in \tilde{L}^s_t \cap A(r_1, (s+t)^{-1/8}),$$

and if in addition $r \leq 2$, then

$$\tilde{\Theta}^s_t(y_0, r) \leq 1 + \frac{\varepsilon_0}{2} + \nu \qquad \forall y_0 \in A(r_1, (s+t)^{-1/8}).$$

Remark 5.4.6. *Note in particular that r_1 does not depend on ν, which will be important later.*

Proof. We consider $t \leq \delta_2$ and $s \leq s_2$ (both δ_2 and s_2 to be chosen) and define

$$l := \frac{t}{2(s+t)} \qquad \Sigma^{(s,t)} := \frac{L^s}{\sqrt{2(s+t)}}.$$

Clearly $l \leq 1/2$ and also from (H4) we have that if s_2, δ_2 are chosen small enough, then $\Sigma^{(s,t)} \cap A(r_0, 3(s+t)^{-1/8})$ is graphical over $P \cap A(r_0, 3(s+t)^{-1/8})$. Moreover if $v_{(s,t)}$ is the function arising from this graphical decomposition then we have by scaling the estimate of (H4) that

$$|v_{(s,t)}(x)| + |x||\overline{\nabla} v_{(s,t)}(x)| + |x|^2|\overline{\nabla}^2 v_{(s,t)}(x)|$$

$$
\begin{aligned}
&\leq D_3 \left(\sqrt{2(s+t)}|x|^2 + \left(\frac{\sqrt{2s}}{\sqrt{2(s+t)}} \right) e^{-2b(s+t)|x|^2/2s} \right) \\
&\leq D_3 \left(\sqrt{2(s+t)}|x|^2 + e^{-b|x|^2} \right).
\end{aligned}
$$

Let $c > 0$ be a constant that will be chosen later. If $s_2(D_3, r_0, c)$ and $\delta_2(D_3, r_0, c) > 0$ are small enough and $r_1(P, c) \geq \max\{r_0, 1\}$ is chosen to be large enough then we can ensure that

$$
|v_{(s,t)}(x)| + |x||\overline{\nabla} v_{(s,t)}(x)| \leq D_3 \left(\sqrt{2(s+t)}|x|^2 + e^{-b|x|^2} \right) \leq c/2 \tag{5.4.4}
$$

on $A(r_1, 3(s+t)^{-1/8})$. Indeed $x \in A(r_1, 3(s+t)^{-1/8})$ implies that $|x|^2 \leq 9(s+t)^{-1/4}$, and so $\sqrt{2(s+t)}|x|^2 \leq 9\sqrt{2}(s_2 + \delta_2)^{1/4}$ can be bounded in terms of s_2 and δ_2. From now on we fix some $y_0 \in \tilde{L}_t^s \cap A\left(3r_1 + 1, (s+t)^{-1/8}\right)$. Since $y_0\sqrt{2(s+t)}$ is a regular point of (L_t^s), the monotonicity formula implies

$$
1 \leq \Theta_0^s(y_0\sqrt{2(s+t)}, \sqrt{t}) = \int_{\Sigma^{(s,t)}} \Phi(y_0, l) \mathrm{d}\mathcal{H}^n =: I + J + K,
$$

where

$$
\begin{aligned}
I &:= \int_{\Sigma^{(s,t)} \setminus B_{3(s+t)^{-1/8}}} \Phi(y_0, l) \mathrm{d}\mathcal{H}^n, \\
J &:= \int_{\Sigma^{(s,t)} \cap B_{r_1}} \Phi(y_0, l) \mathrm{d}\mathcal{H}^n, \\
K &:= \int_{\Sigma^{(s,t)} \cap A(r_1, 3(s+t)^{-1/8})} \Phi(y_0, l) \mathrm{d}\mathcal{H}^n.
\end{aligned}
$$

We first estimate I. If $|x| \geq 3(s+t)^{-1/8} \geq 3|y_0|$ then

$$
|x - y_0|^2 \geq |x|^2 - 2|x||y_0| + |y_0|^2 \geq |x|^2 - \frac{2|x|^2}{3} + |y_0|^2 = \frac{|x|^2}{3} + |y_0|^2,
$$

so

$$
\Phi(y_0, l) = \frac{1}{(4\pi l)^{n/2}} e^{-|x-y_0|^2/4l} \leq \frac{1}{(4\pi l)^{n/2}} e^{-|y_0|^2/4l} e^{-|x|^2/12l} = 3^{n/2} e^{-|y_0|^2/4l} \Phi(0, 3l).
$$

Therefore by choosing $C_1 = C_1(D_1, n)$ we can estimate

$$\begin{aligned} I = \int_{\Sigma^{(s,t)} \setminus B_{3(s+t)^{-1/8}}} \Phi(y_0, l) \mathrm{d}\mathcal{H}^n &\leq 3^{n/2} e^{-|y_0|^2/4l} \int_{\Sigma^{(s,t)} \setminus B_{3(s+t)^{-1/8}}} \Phi(0, 3l) \mathrm{d}\mathcal{H}^n \\ &\leq 3^{n/2} e^{-|y_0|^2/4l} \int_{(3l)^{-1/2}\Sigma^{(s,t)}} \Phi(0, 1) \mathrm{d}\mathcal{H}^n \\ &\leq C_1 e^{-|y_0|^2/C_1}, \end{aligned}$$

since l is bounded independent of s and t, and the estimate (H1) is scale invariant, so in particular is satisfied by $(3l)^{-1/2}\Sigma^{(s,t)}$.

Next we estimate J. Similarly as before we find that for $|x| \leq r_1 \leq |y_0|/3$ we have

$$|x - y_0|^2 \geq |x|^2 + \frac{|y_0|^2}{3}.$$

Thus

$$\Phi(y_0, l) \leq e^{-|y_0|^2/12l} \Phi(0, l) \qquad \text{on } B_{r_1},$$

hence by possibly increasing C_1 if necessary we have

$$J = \int_{\Sigma^{(s,t)} \cap B_{r_1}} \Phi(y_0, l) \mathrm{d}\mathcal{H}^n \leq e^{-|y_0|^2/12l} \int_{\Sigma^{(s,t)} \cap B_{r_1}} \Phi(0, l) \mathrm{d}\mathcal{H}^n \leq C_1 e^{-|y_0|^2/C_1}.$$

Finally we deal with K. We denote by a_i the orthogonal projection of y_0 onto P_i and by b_i the orthogonal projection of y_0 onto $P_i^\perp$. We suppose without loss of generality that

$$\operatorname{dist}(y_0, P) = |b_1|.$$

We will also denote by $\Sigma_i^{(s,t)}$ the component of $\Sigma^{(s,t)} \cap A(r_1, 3(s+t)^{-1/8})$ that is graphical over $\Pi_i := P_i \cap A(r_1, 3(s+t)^{-1/8})$, and by $v^i_{(s,t)}$ the corresponding graph function. Since $P_1 \cap P_2 = \{0\}$ it follows that for some $c = c(P) > 0$ we have that $|b_2| \geq c|y_0|$. Notice that since $|b_2| \leq |y_0|$ we have that $c \leq 1$. Suppose that $x \in \Sigma_2^{(s,t)}$, and denote by x' the orthogonal projection onto P_2. Then we have

$$|y_0 - x|^2 = |a_2 + b_2 - x' - v^2_{(s,t)}(x')|^2 = |a_2 - x'|^2 + |b_2 - v^2_{(s,t)}(x')|^2.$$

Moreover by (5.4.4), if r_1 is chosen large enough (in particular larger than 1),

$$|v^2_{(s,t)}(x')| \leq \frac{c}{2} \leq \frac{c|y_0|}{2},$$

so

$$|b_2 - v^2_{(s,t)}(x')| \geq |b_2| - |v^2_{(s,t)}(x')| \geq \frac{c|y_0|}{2}.$$

Consequently, denoting by $g_{ij} := \delta_{ij} + D_i v^2_{(s,t)} \cdot D_j v^2_{(s,t)}$ the induced metric on the graph, we can estimate

$$\begin{aligned}
&\int_{\Sigma_2^{(s,t)}} \Phi(y_0,l)\mathrm{d}\mathcal{H}^n \\
&\quad = \int_{\Pi_2} \frac{1}{(4\pi l)^{n/2}} \exp\left(\frac{-|a_2 - x'|^2 - |b_2 - v^2_{(s,t)}(x')|^2}{4l}\right)\sqrt{\det(g_{ij})}\mathrm{d}x' \\
&\quad \leq Ce^{-c^2|y_0|^2/16l} \int_{P_2} \frac{1}{(4\pi l)^{n/2}} e^{-|a_2 - x'|^2/4l}\mathrm{d}x' \\
&\quad \leq C_1 e^{-|y_0|^2/C_1},
\end{aligned}$$

where we used (5.4.4) to estimate the gradient terms arising in the surface measure. Combining this with the estimates for I and J we have that

$$1 \leq \int_{\Sigma^{(s,t)}} \Phi(y_0,l)\mathrm{d}\mathcal{H}^n \leq \int_{\Sigma_1^{(s,t)}} \Phi(y_0,l)\mathrm{d}\mathcal{H}^n + C_1 \exp\left(\frac{-|y_0|^2}{C_1}\right). \tag{5.4.5}$$

Increasing r_1 for the last time if necessary, we can ensure that

$$C_1 \exp\left(\frac{-|y_0|^2}{C_1}\right) \leq \frac{1}{2}.$$

Therefore we have that

$$\frac{1}{2} \leq \int_{\Sigma_1^{(s,t)}} \Phi(y_0,l)\mathrm{d}\mathcal{H}^n \leq C \sup_{\Pi_1} \exp\left(-\frac{|b_1 - v^1_{(s,t)}|^2}{4l}\right).$$

Therefore it follows that $|b_1 - v^1_{(s,t)}|^2/4l$ is bounded on Π_1 independently of l, s and t, thus we can estimate

$$\frac{|b_1 - v^1_{(s,t)}|^2}{4l} \leq C\left(1 - e^{-|b_1 - v^1_{(s,t)}|^2/4l}\right),$$

on Π_1 where C is independent of s and t. Moreover because $(D_i v^1_{(s,t)} \cdot D_j v^1_{(s,t)})$ has non-negative eigenvalues we have that $\sqrt{\det(g_{ij})} \geq 1$, so we can estimate

$$\int_{\Pi_1} \frac{|v^1_{(s,t)} - b_1|^2}{4l} \frac{\exp(-|x' - a_1|^2/4l)}{(4\pi l)^{n/2}}\mathrm{d}x'$$

$$
\begin{aligned}
&\leq C\int_{\Pi_1}\left(1-\exp\left(-\frac{|v^1_{(s,t)}-b_1|^2}{4l}\right)\right)\frac{\exp(-|x'-a_1|^2/4l)}{(4\pi l)^{n/2}}\sqrt{\det(g_{ij})}\mathrm{d}x' \\
&= C\left(\int_{\Pi_1}\frac{\exp(-|x'-a_1|^2/4l)}{(4\pi l)^{n/2}}\sqrt{\det(g_{ij})}\mathrm{d}x' - \int_{\Sigma_1^{(s,t)}}\Phi(y_0,l)\mathrm{d}\mathcal{H}^n\right) \\
&\leq C\left(\int_{\Pi_1}\frac{\exp(-|x'-a_1|^2/4l)}{(4\pi l)^{n/2}}\sqrt{\det(g_{ij})}\mathrm{d}x' - 1\right) + C_1\exp(-|y_0|^2/C_1) \\
&\leq C\int_{\Pi_1}\frac{\exp(-|x'-a_1|^2/4l)}{(4\pi l)^{n/2}}\left(\sqrt{\det(g_{ij})}-1\right)\mathrm{d}x' + C_1\exp(-|y_0|^2/C_1),
\end{aligned}
$$

where we used (5.4.5). We have $\sqrt{1+x} = 1 + x/2 + O(x^2)$ and $\det(I+A) = 1+\mathrm{tr}(A)+O(|A|^2)$, which follows from the Taylor expansions for square root and determinant, hence

$$
\begin{aligned}
\sqrt{\det(g_{ij})} - 1 &= \left(1+\sum_{i=1}^{n}|D_i v^1_{(s,t)}|^2 + O(|\overline{\nabla} v^1_{(s,t)}|^4)\right)^{1/2} - 1 \\
&\leq \frac{n}{2}|\overline{\nabla} v^1_{(s,t)}|^2 + O(|\overline{\nabla} v^1_{(s,t)}|^4) \\
&\leq C|\overline{\nabla} v^1_{(s,t)}|^2,
\end{aligned}
$$

where the last line follows from the fact that $|\overline{\nabla} v^1_{(s,t)}|$ is bounded on $A(r_1, 3(s+t)^{-1/8})$ by (5.4.4). Putting the above two estimates together we find

$$
\begin{aligned}
&\int_{\Pi_1}\frac{|v^1_{(s,t)}-b_1|^2}{4l}\frac{\exp(-|x'-a_1|^2/4l)}{(4\pi l)^{n/2}}\mathrm{d}x' \\
&\qquad\leq C\int_{\Pi_1}|\overline{\nabla} v^1_{(s,t)}|^2\frac{\exp(-|x'-a_1|^2/4l)}{(4\pi l)^{n/2}}\mathrm{d}x' + C_1\exp(-|y_0|^2/C_1).
\end{aligned}
$$

Therefore since

$$
|b_1|^2 \leq (|b_1 - v^1_{(s,t)}| + |v^1_{(s,t)}|)^2 \leq 2(|b_1 - v^1_{(s,t)}|^2 + |v^1_{(s,t)}|^2),
$$

we can estimate, by integrating both sides against $(4\pi l)^{n/2}\exp(-|x'-a_1|^2/4l)$ over Π_1

$$
|b_1|^2 \leq C_1\int_{\Pi_1}(|v^1_{(s,t)}|^2 + |\overline{\nabla} v^1_{(s,t)}|)\frac{\exp(-|x'-a_1|^2/4l)}{(4\pi l)^{n/2}}\mathrm{d}x' + C_1\exp(-|y_0|^2/C_1). \tag{5.4.6}
$$

Note that here we used the fact that the integral of $(4\pi l)^{n/2}\exp(-|x'-a_1|^2/4l)$ over Π_1 can be bounded below by a constant, on account of the fact that l is bounded independently of s and t, and the outer radius in the definition of Π_1 is bounded below by $3(s_2+\delta_2)^{-1/8}$ which, by choice of s_2 and δ_2, we can assume to be greater than $2r_1$ say. Since b_1 is also constant we rearrange to obtain the above identity. We want to now control the integral terms on the right hand side. First we observe that $|a_1| \geq c|y_0|$ for some constant depending only on P. This follows from the fact that we assumed y_0 was closer to P_1 than P_2, and hence lies in some fixed conical neighbourhood of P_1. Moreover for any $0 \leq l \leq 1$ we have for any x, $a_1 \in \mathbb{R}^{2n}$

$$\begin{aligned} 2b|x+a_1|^2 + \frac{|x|^2}{4l} &= |x|^2\left(\frac{1}{4l}+2b\right) + 2|a_1|^2 b + 4bx\cdot a_1 \\ &\geq |x|^2\left(\frac{1}{4l}+2b\right) + 2|a_1|^2 b - \frac{16bl+1}{8l}|x|^2 - \frac{32b^2 l}{16bl+1}|a_1|^2 \\ &\geq \frac{|x|^2}{8l} + \frac{2b|a_1|^2}{16bl+1}. \end{aligned}$$

Furthermore for $x \in \Pi_1$ we have $|x| \geq 1$, so by (5.4.4)

$$|\overline{\nabla} v^1_{(s,t)}|^2 \leq |x|^2 |\overline{\nabla} v^1_{(s,t)}|^2 \leq C\left((s+t)|x|^2 + e^{-2b|x|^2}\right).$$

Hence for some $C_1 = C_1(D_1, D_3, P)$ we have

$$\begin{aligned} \int_{\Pi_1} |\overline{\nabla} v^1_{(s,t)}|^2 \frac{e^{-|x'-a_1|^2/4l}}{(4\pi l)^{n/2}}\mathrm{d}x' &\leq C_1 \int_{\Pi_1}\left((s+t)|x'|^2 + e^{-2b|x'|^2}\right)\frac{e^{-|x'-a_1|^2/4l}}{(4\pi l)^{n/2}}\mathrm{d}x' \\ &\leq C_1(s+t) + C_1\int_{\mathbb{R}^n} e^{-b|x'|^2}\frac{e^{-|x'-a_1|^2/4l}}{(4\pi l)^{n/2}}\mathrm{d}x' \\ &\leq C_1(s+t) + C_1\int_{\mathbb{R}^n} e^{-b|x'+a_1|^2}\frac{e^{-|x'|^2/4l}}{(4\pi l)^{n/2}}\mathrm{d}x' \\ &\leq C_1(s+t) + C_1 e^{-|a_1|^2/C_1}\int_{\mathbb{R}^n} \frac{e^{-|x'|^2/8l}}{(4\pi l)^{n/2}}\mathrm{d}x' \\ &\leq C_1\left((s+t) + e^{-|y_0|^2/C_1}\right). \end{aligned}$$

Here we used the fact that integrating $|x'|^2$ against $(4\pi l)^{-n/2}\exp(-|x'-a_1|^2/4l)$ over $\mathbb{R}^n$ can be bounded in terms of the scale l, which is itself bounded by $1/2$. Similarly, using (5.4.4) again, we can estimate

$$|v^1_{(s,t)}|^2 \leq C_1\left((t+s)|x|^4 + e^{-2b|x|^2}\right).$$

So an entirely analogous calculation establishes the estimate

$$\int_{\Pi_1} |v^1_{(s,t)}|^2 \frac{e^{-|x'-a_1|^2/4l}}{(4\pi l)^{n/2}} \mathrm{d}x' \leq C_1 \left((s+t) + e^{-|y_0|^2/C_1} \right).$$

Therefore from (5.4.6) we have

$$|b_1|^2 \leq C_1 \left((s+t) + e^{-|y_0|^2/C_1} \right),$$

so choosing s_2 and δ_2 depending on D_1, D_2, P, r_0, ν and b we have that for all $s \leq s_2$ and $t \leq \delta_2$ we have

$$|b_1| = \mathrm{dist}(y_0, P) \leq \nu + C_1 e^{-|y_0|^2/C_1}.$$

We next want to show that by possibly increasing r_1, and decreasing s_1 and δ_1 if necessary, that we also have the estimate

$$\tilde{\Theta}^s_t(y_0, r) \leq 1 + \frac{\varepsilon}{2} + \nu$$

for any $r \leq 2$. We have

$$\begin{aligned}
\tilde{\Theta}^s_t(y_0, r) &= \int_{\tilde{L}^s_t} \frac{1}{(4\pi r^2)^{n/2}} \exp\left(\frac{-|x - y_0|^2}{4r^2} \right) \mathrm{d}\mathcal{H}^n \\
&= \int_{L^s_t} \frac{1}{(4\pi (2(s+t)) r^2)^{n/2}} \exp\left(\frac{-|x - \sqrt{2(s+t)} y_0|^2}{4r^2(2(s+t))} \right) \mathrm{d}\mathcal{H}^n \\
&= \Theta^s_t(\sqrt{2(s+t)} y_0, \sqrt{2(s+t)} r).
\end{aligned}$$

Applying the monotonicity formula we have

$$\Theta^s_t(\sqrt{2(s+t)} y_0, \sqrt{2(s+t)} r) \leq \Theta^s_0(\sqrt{2(s+t)} y_0, \sqrt{2(s+t) r^2 + t}),$$

so we find, recalling that $l = t/2(s+t)$

$$\begin{aligned}
\tilde{\Theta}^s_t(y_0, r) &\leq \int_{L^s} \frac{1}{(4\pi (2(s+t) r^2 + t))^{n/2}} \exp\left(\frac{-|x - \sqrt{2(s+t)} y_0|^2}{4(2(s+t) r^2 + t)} \right) \mathrm{d}\mathcal{H}^n \\
&= \int_{\Sigma^{(s,t)}} \frac{1}{(4\pi (r^2 + l))^{n/2}} \exp\left(\frac{-|x - y_0|^2}{4(l + r^2)} \right) \mathrm{d}\mathcal{H}^n
\end{aligned}$$

$$= \int_{\Sigma^{(s,t)}} \Phi(y_0, l + r^2) \mathrm{d}\mathcal{H}^n.$$

Therefore by splitting up the integral as before and estimating exactly analogously we have

$$\begin{aligned}
\tilde{\Theta}^s_t(y_0, r) &\leq \int_{\Sigma_1^{(s,t)}} \Phi(y_0, l + r^2) \mathrm{d}\mathcal{H}^n + C_1 \exp\left(\frac{-|y_0|^2}{C_1}\right) \\
&\leq \int_{\Pi_1} \frac{\exp\left(\frac{-|x'-a_1|^2}{4(l+r^2)}\right)}{(4\pi(l+r^2))^{n/2}} \sqrt{\det(g_{ij}}\mathrm{d}x' + C_1 \exp\left(\frac{-|y_0|^2}{C_1}\right) \\
&\leq 1 + C_1 \int_{\Pi_1} |\overline{\nabla} v^1_{(s,t)}|^2 \frac{\exp\left(\frac{-|x'-a_1|^2}{4(l+r^2)}\right)}{(4\pi(l+r^2))^{n/2}} \mathrm{d}x' + C_1 \exp\left(\frac{-|y_0|^2}{C_1}\right) \\
&\leq 1 + C_1(s+t) + C_1 \int_{\mathbb{R}^n} e^{-2b|x'|^2} \frac{\exp\left(\frac{-|x'-a_1|^2}{4(l+r^2)}\right)}{(4\pi(l+r^2))^{n/2}} \mathrm{d}x + C_1 \exp\left(\frac{-|y_0|^2}{C_1}\right) \\
&= 1 + C_1(s+t) + C_1 \int_{\mathbb{R}^n} e^{-2b|x'+a_1|^2} \frac{\exp\left(\frac{-|x'|^2}{4(l+r^2)}\right)}{(4\pi(l+r^2))^{n/2}} \mathrm{d}x + C_1 \exp\left(\frac{-|y_0|^2}{C_1}\right).
\end{aligned}$$

We want to estimate the exponential terms and pull out an exponential factor in $|a_1|$ so we estimate

$$\begin{aligned}
2b|x+a_1|^2 + \frac{|x|^2}{4(l+r^2)} &\geq |x|^2 \frac{8b(l+r^2)+1}{4(l+r^2)} + 2b|a_1|^2 - \frac{16b(l+r^2)+1}{8(l+r^2)}|x|^2 \\
&\quad - \frac{32b^2(l+r^2)}{16b(l+r^2)+1}|a_1|^2 \\
&= \frac{|x|^2}{8(l+r^2)} + \frac{2b|a_1|^2}{16b(l+r^2)+1} \\
&\geq \frac{|x|^2}{8(l+r^2)} + \frac{|a_1|^2}{C_1},
\end{aligned}$$

where we used the fact that l and r are both bounded independently of s and t. Therefore putting this together we have

$$\begin{aligned}
\tilde{\Theta}^s_t(y_0, r) &\leq 1 + C_1(s+t) + C_1 e^{-|a_1|^2/C_1} \int_{\mathbb{R}^n} \frac{e^{-|x|^2/8(l+r^2)}}{(4\pi(l+r^2))^{n/2}} \mathrm{d}x + C_1 e^{-|y_0|^2/C_1} \\
&\leq 1 + C_1(s+t) + C_1 e^{-|y_0|^2/C_1}.
\end{aligned}$$

Evidently an appropriate choice of r_1, s_2 and δ_2 yields the required result. $\square$

The following two Lemmas show that we have additional control in annular regions, specifically on normal deviation, curvature, Lagrangian angle and the primitive for the Liouville form.

Lemma 5.4.7. *Let* $F^s_t \colon L^s \to \mathbb{R}^{2n}$ *be the normal deformation such that* $L^s_t = F^s_t(L^s)$*. We also define* $\tilde{F}^s_t := (2(s+t))^{-1/2} F^s_t$ *so that* $\tilde{L}^s_t = \tilde{F}^s_t(L^s)$*. Then there exist* r_2*,* δ_3*,* s_3 *and* $K < \infty$ *such that if* $t \leq \delta_3$ *and* $s \leq s_3$ *then*

$$\left|\tilde{F}^s_0(x) - \tilde{F}^s_t(x)\right| \leq K \qquad \text{whenever} \qquad \tilde{F}^s_0(x) \in A\left(r_2, (s+t)^{-1/8}/4\right).$$

Proof. By the proximity lemma 5.4.5 we may choose $r_2 \geq 1$, δ_3 and s_3 such that if $t \leq \delta_3$ and $s \leq s_3$ then

$$\Theta^s_t(x, r) \leq 1 + \varepsilon_0$$

for all $r \leq 2\sqrt{2(s+t)}$ and $x \in A\left(r_2\sqrt{2(s+t)}, \sqrt{2}(s+t)^{3/8}\right)$. Hence by White's regularity theorem (Theorem 4.2.6) we can find a C such that

$$\left|\frac{dF^s_t(p)}{dt}\right| = |\vec{H}| \leq \frac{C}{\sqrt{t}},$$

whenever $F^s_t(p) \in A\left(2r_2\sqrt{2(s+t)}, \sqrt{2(s+t)}(s+t)^{-1/8}/2\right)$. Therefore, choosing a larger r_2 and smaller s_3, δ_3 if necessary we obtain, by the fundamental theorem of calculus,

$$|F^s_t(p) - F^s_0(p)| \leq \int_0^t \frac{C}{\sqrt{s}} \mathrm{d}s = 2C\sqrt{t},$$

whenever

$$F^s_0(p) \in A\left(r_2(2(s+t))^{1/2}, (2(s+t))^{1/2}(s+t)^{-1/8}/4\right),$$

which establishes the result. ☐

Lemma 5.4.8. *There are* $\delta_4 > 0$ *and* $s_4 > 0$ *such that for* $0 < s \leq s_4$ *and* $t < \delta_4$

$$|A^s_t(x)| + |\theta^s_t(x)| + |\beta^s_t(x)| \leq D_4 \qquad \forall x \in L^s_t \cap A(1/3, 3). \tag{5.4.7}$$

Proof. The estimate is clearly true for $t = 0$ by assumption (H2). Moreover, by (H4) we can assume that for s sufficiently small, each of the L^s is the graph of a function with small gradient in the region $A(1/4, 4)$. Applying Lemma 5.7.1 we find that L^s remains graphical with small gradient in $A(2/7, 7/2)$ for some short time, which implies that $|\theta^s_t| \leq C$ for δ_4 chosen small enough.

That $|A_t^s|$ is bounded follows from Lemma 5.7.1 and Corollary 5.7.4, since Lemma 5.7.1 implies small gradient for a short time, which allows us to apply Corollary 5.7.4 to get uniform curvature bounds for some short time in $A(1/3, 3)$.

Since $|\theta_t^s|$ and $|A_t^s|$ are both bounded, we have from the evolution equations of β_t^s (see Lemma 5.2.1) that

$$\left|\frac{d\beta_t^s}{dt}\right| \leq \left|\langle Jx, \vec{H}\rangle\right| + 2|\theta_t^s| \leq C.$$

Hence for some suitable short time, $|\beta_t^s|$ also remains bounded in $A(1/3, 3)$. □

The last of the technical lemmas in this section uses the monotonicity formula of Section 5.2 to show that after waiting for a short time dependent on s, we can find times at which the scaled flow $\tilde{L}_t^s$ is close to a self-expander in an L^2 sense. We later use this in the proof of the main theorem to get estimates on the density ratios via the stability result.

Lemma 5.4.9. *Let $a > 1$. Let q_1 be as given by Lemma 5.4.4, and set $q := q_1/a$. Then for all $\eta > 0$ and $R > 0$ there exist $\delta_5 > 0$, $s_5 > 0$ such that for all $s \leq s_5$ and $qs \leq T \leq \delta_5$ we have*

$$\frac{1}{(a-1)T}\int_T^{aT}\int_{\tilde{L}_t^s \cap B_R} |\vec{H} - x^\perp|^2 \mathrm{d}\mathcal{H}^n \mathrm{d}t \leq \eta.$$

Proof. Fix $R > 0$, $\eta > 0$. Suppose $s \leq s_5$ and $qs \leq T \leq \delta_5$, with δ_5 and s_5 yet to be determined. Furthermore, we set $T_0 := R^2(s + aT) + aT$. Throughout the proof, we denote by C a constant which depends on a, R and q, but not on T or s. We estimate

$$\begin{aligned}\frac{1}{(a-1)T}&\int_T^{aT}\int_{\tilde{L}_t^s \cap B_R} |\vec{H} - x^\perp|^2 \mathrm{d}\mathcal{H}^n \mathrm{d}t \\ &= \frac{1}{(a-1)T}\int_T^{aT}(2(s+t))^{-n/2-1}\int_{L_t^s \cap B_{R\sqrt{2(s+t)}}} |2(s+t)\vec{H} - x^\perp|^2 \mathrm{d}\mathcal{H}^n \mathrm{d}t.\end{aligned} \tag{5.4.8}$$

We can ensure $R\sqrt{2(s+t)} \leq 2$ if we choose s_5 and δ_5 small enough. Moreover on $B_{R\sqrt{2(s+t)}}$ we have

$$(T_0-t)^{n/2}\rho_{0,T_0}(x,t) = \frac{1}{(4\pi)^{n/2}}\exp\left(-\frac{|x|^2}{4(T_0-t)}\right)$$
$$\geq \frac{1}{(4\pi)^{n/2}}\exp\left(-\frac{R^2 2(s+t)}{4(T_0-t)}\right).$$

Since $T_0 - t = R^2(s+aT) + aT - t \geq R^2(s+aT) \geq R^2(s+t)$, it follows that

$$(T_0-t)^{n/2}\rho_{0,T_0}(x,t) \geq \frac{1}{(4\pi)^{n/2}}\exp\left(-\frac{1}{2}\right).$$

Hence we can continue estimating (5.4.8) using the localized monotonicity formula of Lemma 5.2.4 (ϕ denotes the cut-off function given in that lemma which is 1 on B_2 and 0 outside of B_3)

$$\begin{aligned}(5.4.8) &\leq \frac{C}{T}\int_T^{aT}(s+t)^{-(n+2)/2}(T_0-t)^{n/2}\int_{L_t^s}\phi|2(s+t)\vec{H}-x^{\perp}|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n\mathrm{d}t\\ &\leq \frac{C}{T}\int_T^{aT}(s+T)^{-(n+2)/2}(T_0-T)^{n/2}\int_{L_t^s\cap A(2,3)}|\beta_t^s+2(s+t)\theta_t^s|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n\mathrm{d}t\\ &\quad+\frac{C}{T}(s+T)^{-(n+2)/2}(T_0-T)^{n/2}\int_{L_T^s}\phi|\beta_T^s+2(s+T)\theta_T^s|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n.\end{aligned} \tag{5.4.9}$$

Now using the localized monotonicity a second time we have the estimate

$$\frac{d}{dt}\int_{L_t^s}\phi|\beta_t^s+2(s+t)\theta_t^s|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n \leq C\int_{L_t^s\cap A(2,3)}|\beta_t^s+2(s+t)\theta_t^s|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n$$

so

$$\begin{aligned}\int_{L_T^s}\phi|\beta_T^s+2(s+T)\theta_T^s|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n &\leq \int_{L_0^s}\phi|\beta_0^s+2s\theta_0^s|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n\\ &\quad+C\int_0^T\int_{L_t^s\cap A(2,3)}|\beta_t^s+2(s+t)\theta_t^s|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n\mathrm{d}t,\end{aligned}$$

hence

$$\begin{aligned}(5.4.9) &\leq \frac{C}{T}(s+T)^{-(n+2)/2}(T_0-T)^{n/2}\int_{L_0^s}\phi|2s\theta_0^s+\beta_0^s|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n\\ &\quad+\frac{C}{T}(s+T)^{-(n+2)/2}(T_0-T)^{n/2}\int_0^{aT}\int_{L_t^s\cap A(2,3)}|2(s+t)\theta_t^s+\beta_t^s|^2\rho_{0,T_0}\mathrm{d}\mathcal{H}^n\mathrm{d}t.\end{aligned}$$

Now $T_0 - T \le C(s+T)$, with C depending only on R and a, so estimating the terms in front of the integrals we have

$$\begin{aligned}
(5.4.9) \le & \frac{C}{T(s+T)} \int_{L_0^s} \phi |2s\theta_0^s + \beta_0^s|^2 \rho_{0,T_0} \mathrm{d}\mathcal{H}^n \\
& + \frac{C}{T(s+T)} \int_0^{aT} \int_{L_t^s \cap A(2,3)} |2(s+t)\theta_t^s + \beta_t^s|^2 \rho_{0,T_0} \mathrm{d}\mathcal{H}^n \mathrm{d}t \\
=: & \, A + B.
\end{aligned}$$

We first estimate B. Notice that by Lemma 5.4.8 we have

$$|2(s+t)\theta_t^s + \beta_t^s|^2 \le (2(s+t)|\theta_t^s| + |\beta_t^s|)^2 \le C((s+t)+1)^2.$$

Hence, we can estimate

$$\begin{aligned}
B &\le \frac{C((s+aT)+1)^2}{T(s+T)} \int_0^{aT} \int_{L_t^s \cap A(2,3)} \rho_{0,T_0} \mathrm{d}\mathcal{H}^n \mathrm{d}t \\
&\le \frac{C((s+aT)+1)^2}{T(s+T)} \int_0^{aT} \int_{L_t^s \cap A(2,3)} |x|^4 \rho_{0,T_0} \mathrm{d}\mathcal{H}^n \mathrm{d}t \\
&= \frac{C((s+aT)+1)^2}{T(s+T)} \int_0^{aT} (T_0 - t)^2 \int_{(T_0-t)^{-1/2}(L_t^s \cap A(2,3))} |x|^4 \rho_{0,1} \mathrm{d}\mathcal{H}^n \mathrm{d}t \\
&\le \frac{C((s+aT)+1)^2}{T(s+T)} T_0^3 \sup_{t \in [0,aT]} \int_{(T_0-t)^{-1/2}(L_t^s \cap A(2,3))} |x|^4 \exp\left(-\frac{|x|^2}{4}\right) \mathrm{d}\mathcal{H}^n.
\end{aligned} \tag{5.4.10}$$

We note that $T_0 \le (R^2(1/q+a)+a)T = CT$, $T_0 \le C(s+T)$ and $T_0 \ge R^2(s+aT)$ so we can estimate

$$\begin{aligned}
(5.4.10) &\le C(T_0+1)^2 T_0 \sup_{t \in [0,aT]} \int_{(T_0-t)^{-1/2} L_t^s \cap A(2,3)} |x|^4 \exp\left(-\frac{|x|^2}{4}\right) \mathrm{d}\mathcal{H}^n \\
&\le C(T_0+1)^2 T_0,
\end{aligned}$$

where we can estimate the supremum by a uniform constant because L_t^s all have bounded area ratios with a uniform constant. Moreover $T_0 \le R^2\delta_5(1/q+a) + a\delta_5$ so that by possibly decreasing δ_5 we can ensure that $B \le \eta/2$.

We next estimate A,

$$A \le \frac{C}{T(s+T)} \int_{L_0^s \cap B_3} |2s\theta_0^s + \beta_0^s|^2 \rho_{0,T_0} \mathrm{d}\mathcal{H}^n \mathrm{d}t. \tag{5.4.11}$$

First recall that if β^s is primitive for the Liouville form on some L^s, then $\beta^s_l := l^{-2}\beta^s$ is primitive for the Liouville form on $l^{-1}L^s$. From here on we surpress the subscript 0 of the β^s and θ^s since we only ever integrate over the manifolds L^s_0, and we instead use a subscript l to denote the rescaling factor of the β^s. We define

$$l := \sqrt{2(s+T)} \qquad \sigma := \frac{s}{s+T}$$

then

$$\begin{aligned}(5.4.11) &= \frac{C(s+T)}{T}\int_{l^{-1}(L^s_0\cap B_3)} |\sigma\theta^s + \beta^s_l|^2 \rho_{0,l^{-2}T_0} \mathrm{d}\mathcal{H}^n \mathrm{d}t \\ &\leq C\int_{l^{-1}(L^s_0\cap B_3)} |\sigma\theta^s + \beta^s_l|^2 \rho_{0,l^{-2}T_0} \mathrm{d}\mathcal{H}^n,\end{aligned}$$

since $T \geq qs$, so we can absorb $(s+T)/T$ into the constant. Define

$$F(s,T) := \int_{l^{-1}(L^s_0\cap B_3)} |\sigma\theta^s + \beta^s_l|^2 \rho_{0,l^{-2}T_0} \mathrm{d}\mathcal{H}^n.$$

Notice that from the definition of T_0 we can find $C > 0$ independent of T and s such that $l^{-2}T_0 \in [C^{-1}, C]$. We want to show that by possibly again decreasing s_5 and δ_5, we can ensure $F(s,T) \leq \eta/2$. Seeking a contradiction, suppose that this is not the case. Then we can find sequences s_i and T_i both converging to 0 with $qs_i \leq T_i$ and such that $F(s_i, T_i) > \eta/2$. After possibly extracting a subsequence which we don't relabel, we may assume that $l_i^{-2}T_0 \to T_1$. We split the rest of the proof into two cases.

Case 1: Suppose that (after possibly extracting a further subsequence) we have that $\sigma_i \to \sigma > 0$. Then by (H3) we have

$$l_i^{-1}L^{s_i}_0 = \sigma_i^{1/2}\tilde{L}^{s_i}_0 \to \sigma^{1/2}\Sigma$$

in $C^{1,\alpha}$. Therefore we have

$$\begin{aligned}\lim_{i\to\infty} F(T_i, s_i) &= \lim_{i\to\infty}\int_{\sigma_i^{1/2}\tilde{L}^{s_i}_0 \cap l_i^{-1}B_3} |\sigma_i\theta^{s_i} + \beta^{s_i}_{l_i}|^2 \rho_{0,l_i^{-2}T_0} \mathrm{d}\mathcal{H}^n \\ &= \lim_{i\to\infty}\sigma_i^2\int_{\tilde{L}^{s_i}_0\cap(2s_i)^{-1/2}B_3} |\tilde{\theta}^{s_i} + \tilde{\beta}^{s_i}|^2 \rho_{0,l_i^{-2}\sigma_i^{-1}T_0} \mathrm{d}\mathcal{H}^n = 0,\end{aligned}$$

because $|\tilde{\theta}^{s_i}+\tilde{\beta}^{s_i}|$ is bounded by $D_2(1+|x|^2)$ on $B_{3(2s_i)^{-1/2}}$, which means that since $l_i^{-2}\sigma_i^{-1}T_0 \to \sigma^{-1}T_1 > 0$ the contribution to the integral outside some fixed large ball is small uniformly in i. Moreover by (H3) we have $\lim_{i\to\infty}|\tilde{\theta}^{s_i}+\tilde{\beta}^{s_i}|^2 = 0$ locally, so inside this large ball the integral can be made as small as desired.

Case 2: Suppose now that after possibly passing to subsequence, which we do not relabel, we have $\sigma_i \to 0$. Then, with r_0 defined as in property (H4) of the family L^s, we find

$$\begin{aligned}\lim_{i\to\infty}\int_{l_i^{-1}(L_0^{s_i}\cap B_{r_0\sqrt{s_i}})}&|\sigma_i\theta^{s_i}+\beta_{l_i}^{s_i}|^2\rho_{0,l_i^{-2}T_0}\mathrm{d}\mathcal{H}^n\\ &=\lim_{i\to\infty}\int_{\sigma_i^{1/2}\tilde{L}_0^{s_i}\cap B_{r_0\sqrt{\sigma_i/2}}}|\sigma_i\theta^{s_i}+\beta_{l_i}^{s_i}|^2\rho_{0,l_i^{-2}T_0}\mathrm{d}\mathcal{H}^n\\ &=\lim_{i\to\infty}\sigma_i^2\int_{\tilde{L}_0^{s_i}\cap B_{r_0/\sqrt{2}}}|\tilde{\theta}^{s_i}+\tilde{\beta}^{s_i}|^2\rho_{0,\sigma_i^{-1}l_i^{-1}T_0}\mathrm{d}\mathcal{H}^n=0,\end{aligned}$$

because $|\tilde{\theta}^{s_i}+\tilde{\beta}^{s_i}|^2 \to 0$ locally, and ρ is bounded. So to estimate $\lim_{i\to\infty}F(T_i,s_i)$ we need only control the integral in the annulus $A(r_0\sqrt{\sigma_i/2},3l_i^{-1})$. We first notice that by (H4), provided i is large enough, $l_i^{-1}L^{s_i}\cap A(r_0\sqrt{\sigma_i/2},3l_i^{-1})$ is graphical over P, and if v_i is the function arising from this decomposition we have the estimate

$$|v_i(x')|+|x'||\overline{\nabla}v_i(x')|+|x'|^2|\overline{\nabla}^2v_i(x')|\le D_3\left(l_i|x'|^2+\sigma_i^{1/2}e^{-b|x'|^2/2\sigma_i}\right).$$

In the graphical region, the normal space to the graph is spanned by the vectors $n_j := (-\overline{\nabla}v_i^j, e_j)$ for $j=1,\dots,n$ where e_j denotes the vector in $\mathbb{R}^n$ whose jth entry is 1, and all other entries are 0, and v_i^j is the jth coordinate of v_i. Then given an orthonormal basis for the normal space $\nu_1,\dots,\nu_n$ we have $\nu_j=\sum_{k=1}^n\alpha_{jk}n_k$, where α_{jk} are fixed real numbers denoting the coefficients in the basis expansion of ν_j in terms of the n_k. It then follows that

$$|x^\perp|\le C\sum_{j=1}^n|\langle x,n_j\rangle|,$$

where C depends only on the α_{jk}. Now

$$\begin{aligned}\langle x,n_j\rangle&=\langle(x',v_i(x')),(-\overline{\nabla}v_i^j,e_j)\rangle\\ &=-\langle x',\overline{\nabla}v_i^j(x')\rangle+v_i^j(x')\end{aligned}$$

from which it follows that

$$|x^{\perp}| \leq C\left(|v_i(x')| + |x'||\overline{\nabla} v_i(x')|\right).$$

Therefore

$$|\nabla \beta^{s_i}_{l_i}| = |x^{\perp}| \leq C\left(l_i|x'|^2 + \sigma_i^{1/2}\right). \tag{5.4.12}$$

Using this estimate we can control $\beta^{s_i}_{l_i}$ independently of i on the annular region $A(r_0\sqrt{\sigma_i/2}, 3l_i^{-1}) \cap l_i^{-1}L^{s_i}$. Indeed suppose that $x \in A(r_0\sqrt{\sigma_i/2}, 3l_i^{-1}) \cap l_i^{-1}L^{s_i}$, then there is a corresponding $x' \in A(r_0\sqrt{\sigma_i/2}, 3l_i^{-1}) \cap P$ such that $x = x' + v_i(x')$. Define

$$x_i' := \frac{r_0\sqrt{\sigma_i}}{\sqrt{2}}\frac{x'}{|x'|} \quad \text{and} \quad x_i := x_i' + v_i(x_i').$$

Note that x_i of course depends on the original choice of x as well as i. We may now define a curve in $l_i^{-1}L^{s_i}$ by setting

$$\gamma(t) := x_i' + t(x' - x_i') + v_i(x_i' + t(x' - x_i')).$$

By the fundamental theorem of calculus we can write

$$\begin{aligned}\beta^{s_i}_{l_i}(x) &= \beta^{s_i}_{l_i}(x_i) + \int_0^1 \frac{d}{dt}\beta^{s_i}_{l_i}(\gamma(t))\mathrm{d}t \\ &\leq \beta^{s_i}_{l_i}(x_i) + \int_0^1 |\nabla \beta^{s_i}_{l_i}(\gamma(t))||\gamma'(t)|\mathrm{d}t,\end{aligned}$$

and furthermore

$$|\gamma'(t)| \leq |x' - x_i'| + |\overline{\nabla} v_i||x' - x_i'| \leq C|x|$$

so

$$\begin{aligned}\beta^{s_i}_{l_i}(x) &\leq \beta^{s_i}_{l_i}(x_i) + C|x|\int_0^1 l_i|x_i' + t(x' - x_i')|^2 + \sigma_i^{1/2}\mathrm{d}t \\ &\leq \beta^{s_i}_{l_i}(x_i) + C(l_i|x|^3 + \sigma_i^{1/2}).\end{aligned}$$

Now $\beta^{s_i}_{l_i}(x_i) = \sigma_i\tilde{\beta}^{s_i}(\sigma_i^{1/2}x_i)$, moreover since $|x_i|$ is bounded independently of i or the original choice of $|x|$ we have from property (H3) of L^s that

$$\lim_{i\to\infty} \tilde{\beta}^{s_i}(\sigma_i^{1/2}x_i) + \tilde{\theta}^{s_i}(\sigma_i^{1/2}x_i) = 0$$

uniformly in x. Thus

$$\lim_{i\to\infty} \beta^{s_i}_{l_i}(x_i) = -\lim_{i\to\infty} \sigma_i \tilde{\theta}^{s_i}(\sigma_i^{1/2} x_i) = 0$$

uniformly in x as $\tilde{\theta}^{s_i}$ is bounded and $\sigma_i \to 0$. Therefore we may bound the term $\beta^{s_i}_{l_i}(x_i)$ by some sequence b_i with $b_i \to 0$. Consequently we have the estimate

$$|\beta^{s_i}_{l_i}(x)| \leq C\left(l_i|x|^3 + \sigma_i^{1/2}|x|\right) + b_i$$

on $A(r_0\sqrt{\sigma_i/2}, 3l_i^{-1}) \cap l_i^{-1}L^{s_i}$, hence

$$\begin{aligned}
\lim_{i\to\infty} F(T_i, s_i) &= \lim_{i\to\infty} \int_{l_i^{-1}L^{s_i}\cap A(r_0\sqrt{\sigma/2}, 3l_i^{-1})} |\sigma_i\theta^{s_i} + \beta^{s_i}_{l_i}|^2 \rho_{0,l_i^{-2}T_0} \mathrm{d}\mathcal{H}^n \\
&= \lim_{i\to\infty} \int_{l_i^{-1}L^{s_i}\cap A(r_0\sqrt{\sigma/2}, 3l_i^{-1})} |\beta^{s_i}_{l_i}|^2 \rho_{0,l_i^{-2}T_0} \mathrm{d}\mathcal{H}^n \\
&\leq \lim_{i\to\infty} C(l_i^2 + \sigma_i + b_i^2) \int_{l_i^{-1}L^{s_i}} (|x|^6 + |x|^2 + 1)\rho_{0,l_i^{-2}T_0} \mathrm{d}\mathcal{H}^n = 0,
\end{aligned}$$

where we again used the fact that $l_i^{-2}T_0 \to T_1 > 0$, so that outside of some large ball the contribution to the integral is very small. This limit being zero is a contradiction, so we are done. □

We may now embark on the proof of Theorem 5.4.1. Changing scale, to prove the main theorem it would in fact suffice to show the following (which is very slightly stronger due to the bound on the scale of the density ratios),

Theorem (Rescaled main theorem)**.** *There exist s_0, δ_0 and τ such that if $t \leq \delta_0$, $r^2 \leq \tau$ and $s \leq s_0$, then*

$$\tilde{\Theta}^s_t(x_0, r) \leq 1 + \varepsilon_0$$

for all x_0 with $|x_0| \leq (2(s+t))^{-1/2}$.

Let q_1 be defined as in Lemma 5.4.4, and recall that $q_1 < 1$. If we set $\tau := q_1/(2(q_1+1))$, then the rescaled version of Lemma 5.4.4 implies

Lemma (Rescaled short-time existence)**.** *If $s \leq s_1$, $t \leq q_1 s$ and $r^2 \leq \tau$ then*

$$\tilde{\Theta}^s_t(y_0, r) \leq 1 + \varepsilon_0$$

$|y_0| \leq (2(s+t))^{-1/2}$.

Similarly the rescaled Lemma 5.4.2 tells us that

Lemma (Rescaled far from origin)**.** *If* $r^2 \leq \tau$ *and* $q_1 s \leq t \leq \delta_1$

$$\tilde{\Theta}^s_t(y_0, r) \leq 1 + \varepsilon_0$$

whenever $K_0 \leq |y_0| \leq (2(s+t))^{-1/2}$.

Thus to prove the rescaled main theorem, it suffices to show that for appropriately chosen s_0, δ_0 and τ the following holds true: if $r^2 \leq \tau$, $s \leq s_0$, $t \leq \delta_0$ and $t \geq q_1 s$ then

$$\tilde{\Theta}^s_t(y_0, r) \leq 1 + \varepsilon_0$$

whenever $|y_0| \leq K_0$. This is what we now show.

Proof of Theorem 5.4.1. For each s we define

$$T_s := \sup\left\{T \mid \tilde{\Theta}^s_t(y_0, r) \leq 1 + \varepsilon_0 \text{ for all } r^2 \leq \tau,\ t \leq T,\ |y_0| \leq K_0\right\}.$$

We now claim that we can find $\delta_0 > 0$ and $s_0 > 0$ such that $T_s \geq \delta_0$ for all $s \leq s_0$. Indeed, with $\tau = q_1/(2(q_1+1))$ as above, we choose $a > 1$ with $a < (1 + 2\tau)$. Let C_0 be the constant of White's local regularity theorem (Theorem 4.2.6), and set

$$\tilde{C} := C_0 \frac{\sqrt{2(a+3)}}{\sqrt{q_1(a-1)}}. \tag{5.4.13}$$

We next let $r_3 := \max\{r_0, r_1, r_2, 1\}$, where r_0, r_1, and r_2 are as in, respectively, the construction of the approximating family, Lemma 5.4.5, and Lemma 5.4.7. Let $R := \sqrt{1 + 2q_1}K_0 + r_3$ and note that $R \geq 2$. Next fix $\alpha \in (0,1)$ as in the proof of Lemma 5.4.4, and $\varepsilon = \varepsilon(\Sigma, \varepsilon_0, \alpha)$ as given by Lemma 5.7.2. We apply the stability result, Theorem 5.3.3 with $R = R$; $r = r_3$; $C = \max\{C_1, C\}$ the constants from Lemma 5.4.5, and the construction of the approximating family respectively; $M = \tilde{C}$; $\tau = \tau$; $\Sigma = \Sigma$ and $\varepsilon = \varepsilon$. Thus we obtain $\tilde{R} \geq R$, $\eta > 0$ and $\nu \geq 0$ as in the theorem. Apply Lemma 5.4.9 with $\eta = \eta/2$ and $R = \tilde{R}$. This gives s_5 and δ_5 such that the lemma holds. Next apply Lemma 5.4.5 with ν to obtain s_2 and δ_2. We now let $s_0 := \min\{s_1, s_2, s_3, s_4, s_5\}$ and $\delta_0 := \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$.

We finally possibly decrease s_0 and δ_0 slightly to ensure that

$$(s_0 + \delta_0)^{-1/8} \geq 2\tilde{R}.$$

This will ensure that in the annular region $A(r_3, \tilde{R})$ we have all of the estimates of the intermediate lemmas of this section. We now claim that these s_0 and δ_0 are the required constants. Specifically we claim that for all $s \leq s_0$ we have $T_s \geq \delta_0$. Indeed, suppose that this were not the case and that for some $s \leq s_0$ we have $T_s < \delta_0$. Our goal is to show that the hypotheses of Theorem 5.3.3 are satisfied by $\tilde{L}^s_t$ for some t close to T_s, so that we can conclude $\tilde{L}^s_t$ is $C^{1,\alpha}$ close to Σ. Lemma 5.7.2 will then give density ratio bounds for times past T_s, resulting in a contradiction. To this end we define $T := T_s/a$, then since $T < T_s$ we have for all $t \in [T, T_s)$

$$\tilde{\Theta}^s_t(x, r) \leq 1 + \varepsilon_0,$$

for all $r^2 \leq \tau$ and $x \in B_{K_0}$. In fact, as has already been observed, the same is true for all $|x| \leq (s+t)^{-1/8}$, so in particular for all $|x| \leq 2\tilde{R}$. Let $\hat{L}^s_l$ denote the Lagrangian mean curvature flow with initial condition $\tilde{L}^s_T$. Let $\sigma^2 = 2(s+T)$, then we can write $\tilde{L}^s_T = \sigma^{-1} L^s_T$. Then we may write $\hat{L}^s_l$ as

$$\hat{L}^s_l = \sigma^{-1} L^s_{T+\sigma^2 l} = \frac{\sqrt{2(T+s+\sigma^2 l)}}{\sqrt{2(T+s)}} \tilde{L}^2_{T+s+\sigma^2 l} = \sqrt{1+2l}\,\tilde{L}^s_{T+\sigma^2 l}.$$

This implies the density ratio control

$$\hat{\Theta}^s_l(x, r) \leq 1 + \varepsilon_0,$$

for all l such that $T + \sigma^2 l \in [T, T_s)$, $r^2 \leq \tau$ and $x \in B_{2\tilde{R}}$. By White's local regularity theorem (Theorem 4.2.6) we get curvature bounds of the form

$$|\hat{A}^s_l| \leq \frac{C_0}{\sqrt{l}} \qquad l \leq \tau, \text{ on } B_{\tilde{R}},$$

or, scaled back to the original scale this means

$$|A^s_t| \leq \frac{C_0}{\sqrt{t-T}}, \tag{5.4.14}$$

on $B_{\sigma\tilde{R}}$ for all $t < T_s$ with $T \le t \le T + 2(s+T)\tau = (1+2\tau)T + 2\tau s$. Notice in particular that

$$T_s = aT \le (1+2\tau)T + 2\tau s,$$

so the above estimate always holds up to time T_s. Let $t_0 := T(a+1)/2$. Then from (5.4.14) we see

$$|A^s_{t_0}| \le \frac{C_0}{\sqrt{t_0 - T}} = \frac{C_0\sqrt{2}}{\sqrt{(a-1)T}} = \frac{C_0\sqrt{\frac{2(a+3)}{a-1}}}{\sqrt{(a+3)T}} = \frac{C_0\sqrt{\frac{2(a+3)}{a-1}}}{\sqrt{2(t_0+T)}}.$$

Recall that $T \ge q_1 s$ and $q_1 < 1$ so

$$|A^s_{t_0}| \le \frac{C_0\sqrt{\frac{2(a+3)}{a-1}}}{\sqrt{2(t_0+q_1 s)}} \le \frac{\tilde{C}}{\sqrt{2(t_0+s)}},$$

on $B_{\sigma\tilde{R}}$, where $\tilde{C}$ is defined as in (5.4.13). Similarly, if $t > 0$ is such that $t_0 + t \le T_s$ then

$$|A^s_{t_0+t}| \le \frac{C_0}{\sqrt{t_0 + t - T}} \le \frac{C_0\sqrt{2}}{\sqrt{(a-1)T + t}} \le \frac{\tilde{C}}{\sqrt{2(t_0+t+s)}}.$$

In other words, for each $t \in [t_0, T_s)$ we have

$$|A^s_t| \le \frac{\tilde{C}}{\sqrt{2(s+t)}} \quad \text{on } B_{\sigma\tilde{R}},$$

which implies that for each $t \in [t_0, T_s)$ we have

$$|\tilde{A}^s_t| \le \tilde{C} \quad \text{on } B_{\tilde{R}}.$$

This means that $\tilde{L}^s_t$ satisfies condition (i) of Theorem 5.3.3 with $M = \tilde{C}$ and $\tilde{R} = \tilde{R}$ for every $t \in [t_0, T_s)$. Next, applying Lemma 5.4.9, we may select $t_1 \in [t_0, T_s)$ with

$$\int_{\tilde{L}^s_{t_1} \cap B_{\tilde{R}}} |\vec{H} - x^\perp|^2 \mathrm{d}\mathcal{H}^n \le \eta.$$

So $\tilde{L}^s_{t_1}$ also satisfies condition (iii) of Theorem 5.3.3. Condition (iv) of Theorem 5.3.3 holds for $\tilde{L}^s_{t_1}$ by Lemma 5.4.5, and condition (ii) holds by definition of T_s as $t_1 < T_s$. Hence Theorem 5.3.3 implies that $\tilde{L}^s_{t_1}$ is ε-close to Σ in $C^{1,\alpha}(B_{\tilde{R}})$. Redefine $\hat{L}^s_l$ to be the Lagrangian mean curvature flow with initial condition $\tilde{L}^s_{t_1}$.

As before we know that we can write

$$\hat{L}_l^s = \sqrt{1+2l}\tilde{L}^s_{t_1+2(s+t_1)l}.$$

Then Lemma 5.7.2 applied to $\hat{L}_l^s$ says that

$$\hat{\Theta}_l^s(x,r) \leq 1+\varepsilon_0 \qquad r^2,\ l \leq q_1$$

for $|x| \leq \tilde{R}-1$. Since $\tilde{R} \geq R = \sqrt{1+2q_1}K_0 + r_3$ and $r_3 \geq 1$, this means that the same is true for $|x| \leq \sqrt{1+2q_1}K_0$. Rescaling, this is equivalent to

$$\tilde{\Theta}^s_{t_1+2(s+t_1)l}\left(\frac{x}{\sqrt{1+2l}}, \frac{r}{\sqrt{1+2l}}\right) \leq 1+\varepsilon_0,$$

for r^2, $l \leq q_1$ and $|x| \leq \sqrt{1+2q_1}K_0$. Or in other words

$$\tilde{\Theta}_t^s(x,r) \leq 1+\varepsilon_0,$$

for $r^2 \leq q_1/(1+2q_1) = \tau$, $|x| \leq K_0$ and $t_1 \leq t \leq (1+2q_1)t_1 + 2q_1 s$. However, $(1+2q_1)t_1 + 2q_1 s > at_1 > aT = T_s$, which contradicts the definition of T_s. $\square$

5.5 Short-time existence theorem

In this section we prove the following short time existence result using Theorem 5.4.1 and the results of the previous section.

Theorem 5.5.1. *Suppose that $L \subset \mathbb{C}^n$ is a compact Lagrangian submanifold of $\mathbb{C}^n$ with a finite number of singularities, each of which is asymptotic to a pair of transversally intersecting planes $P_1 + P_2$ where neither $P_1 + P_2$ nor $P_1 - P_2$ are area minimizing. Then there exists $T > 0$ and a Lagrangian mean curvature flow $(L_t)_{0<t<T}$ such that as $t \searrow 0$, $L_t \to L$ as varifolds and in C^∞_{loc} away from the singularities.*

Proof. For simplicity we suppose that L has only one singularity at the origin. The case where L has more than one follows by entirely analogous arguments. Recall the one parameter family L^s of section 5.4. Since each L^s is smooth, by standard short time existence theory for smooth compact mean curvature flow, for all $s \in (0,c]$ there exists a Lagrangian mean curvature flow $(L_t^s)_{0\leq t\leq T_s}$

with $T_s > 0$. We claim that there exists a $T_0 > 0$ such that $T_s \geq T_0$ for all s sufficiently small, and that furthermore, we have interior estimates on $|A|$ and its higher derivatives for all $t > 0$, which are independent of s. By virtue of Lemma 5.7.1, we can apply Corollary 5.7.4 on small balls everywhere outside $B_{1/3}$ to get uniform curvature bounds outside of $B_{1/2}$ up to time $\min\{T_s, \delta\}$ where $\delta > 0$ is independent of s. Uniform estimates on the higher derivatives then immediately follow by standard theory of parabolic partial differential equations.

To obtain the desired bounds on $B_{1/2}$ we use Theorem 5.4.1. Let $\varepsilon_0 > 0$ be the constant of Brian White's local regularity theorem. Then Theorem 5.4.1 says that there exist s_0, δ_0 and τ such that for all $s \leq s_0$, $t \leq \delta_0$, $r^2 \leq \tau t$ and $x_0 \in B_{1/2}$ we have

$$\Theta^s_t(x_0, r) = \Theta^s(x_0, t + r^2, r) \leq 1 + \varepsilon_0.$$

This implies that for all $s \leq s_0$, $t \leq \delta_0$ and $r^2 \leq \tau t$ we have $\Theta^s(x_0, t, r) \leq 1 + \varepsilon_0$. We now fix $s \leq s_0$, $t_0 < \min\{\delta_0, T_s\}$, and $\rho \leq \min\{1/4, \sqrt{t_0}\}$. Then it follows that $B_{2\rho}(x_0) \subset B_1$, and furthermore that

$$\Theta^s(x, t, r) \leq 1 + \varepsilon_0$$

for all $r \leq \tau\rho^2$, and $(x, t) \in B_{2\rho}(x_0) \times (t_0 - \rho^2, t_0]$. Then it immediately follows from White's theorem that

$$|A^s_t(x)| \leq \frac{C}{\sqrt{t - t_0 + \rho^2}}$$

for all $(x, t) \in B_\rho(x_0) \times (t_0 - \rho^2, t_0]$, where C depends only on τ and ε_0. These estimates are then uniform in s for $s \leq s_0$. Moreover, these curvature bounds, along with those outside of the ball $B_{1/2}$, imply that $T_s \geq \min\{\delta, \delta_0\}$.

Because the estimates are independent of s, they pass to the limit in the varifold topology when we take a subsequential limit of the flows and so we obtain a limiting flow $(L_t)_{0<t<T_0}$, for which $L_t \to L$ as varifolds.

Note that away from the singularities, we can obtain uniform curvature estimates on $|A|$ thanks to Corollary 5.7.4, so it follows that (L_t) attains the initial data L in C^∞_{loc} away from the singular points. □

5.6 Construction of the approximating family

This section is the result of collaboration with Kim Moore.

In this section, we consider a Lagrangian submanifold L of $\mathbb{C}^n$ with a singularity at the origin which is asymptotic to the pair of planes P considered in Section 5.3. We approximate L by gluing in the self-expander Σ which is asymptotic to P at smaller and smaller scales in place of the singularity. We will show that this yields a family of compact Lagrangians, exact in B_4, which satisfy the hypotheses (H1)-(H4) given in Section 5.4 which are required to implement the analysis in that section.

Since L is conically singular we may write $L \cap B_4$ as a graph over $P \cap B_4$ (possibly rescaling L so that this is the case). We may further apply the Lagrangian neighbourhood theorem (its extension to cones was proved by Joyce, [34, Theorem 4.1]), so that we may identify $L \cap B_4$ with the graph of a one-form γ on P. Recall that the manifold corresponding to the graph of such a one-form is Lagrangian if and only if the one-form is closed.

Moreover, since we have assumed that L is exact inside B_4, there exists $u \in C^\infty(P \cap B_4)$ such that $du = \gamma$. Since we know that γ must decay quadratically, we can choose a primitive for γ which has cubic decay, i.e.,

$$|\nabla^k u(x)| \leq C|x|^{3-k}. \tag{5.6.1}$$

We saw in Theorem 5.3.1 that there exists a unique, smooth zero-Maslov self-expander asymptotic to P. We may also identify the self-expander outside a ball of radius r_0 with the graph of a one-form over P and, since a zero-Maslov class Lagrangian self-expander is globally exact, there exists a function $v \in C^\infty(P \backslash B_{r_0})$ such that the self-expander is described by the exact one-form $\psi = dv$ on $P \backslash B_{r_0}$. Further, Lotay and Neves proved [39, Theorem 3.1]

$$\|v\|_{C^k(P\backslash B_r)} \leq Ce^{-br^2} \quad \text{for all } r \geq r_0. \tag{5.6.2}$$

We will glue $\Sigma_s := \sqrt{2s}\Sigma$ into the initial condition L to resolve the singularity. Our new manifold, L^s, will be the rescaled self-expander Σ^s inside $B_{r_0\sqrt{2s}}$, the manifold L outside B_4 and will smoothly interpolate between the two on the annulus $A(r_0\sqrt{2s}, 4)$.

To do this, we will glue together the primitives of the one-forms corresponding to these manifolds, before taking the exterior derivative. This gives us a one-form that will describe L^s on the annulus $A(r_0\sqrt{2s}, 4)$, which ensures L^s is still Lagrangian and is exact in B_4. We will then show that this family satisfies the properties (H1)-(H4).

Let $\varphi\colon \mathbb{R}_+ \to [0,1]$ be a smooth function satisfying $\varphi \equiv 1$ on $[0,1]$ and $\varphi \equiv 0$ on $[2,\infty)$. Consider the one-form given by, for $r_0\sqrt{2s} \leq |x| \leq 4$, $0 < s \leq c$

$$\gamma_s(x) = dw_s(x) = d\left[\varphi(s^{-1/4}|x|)2sv(x/\sqrt{2s}) + (1-\varphi(s^{-1/4}|x|))u(x)\right], \quad (5.6.3)$$

where we have that $r_0\sqrt{2s} < s^{1/4} < 2s^{1/4} < 4$ holds for all $s \leq c$. Notice that in particular we must have $c < 1$. Then $\gamma_s(x) \equiv \psi_s(x) := \sqrt{2s}\psi(x/\sqrt{2s})$, the one-form corresponding to the rescaled self-expander Σ_s for $|x| < s^{1/4}$ and $\gamma_s \equiv \gamma$ for $|x| > 2s^{1/4}$. Notice that since γ_s is exact, it is closed and therefore its graph corresponds to an exact Lagrangian.

We define the smooth exact Lagrangian L^s by

- $L^s \cap B_{r_0\sqrt{2s}} = \Sigma_s \cap B_{r_0\sqrt{2s}}$,
- $L^s \cap A(r_0\sqrt{2s}, 4) =$graph γ_s,
- $L^s \backslash B_4 = L \backslash B_4$.

We will now show that L^s satisfies (H1)-(H4).

For (H1), notice that both the self-expander and the initial condition individually satisfy (H1), and so for the rescaled self-expander, we have that

$$\begin{aligned}\mathcal{H}^n(\Sigma_s \cap B_R) = \mathcal{H}^n((\sqrt{2s}\Sigma) \cap B_R) &= (2s)^{n/2}\mathcal{H}^n(\Sigma \cap B_{R/\sqrt{2s}}) \\ &\leq (2s)^{n/2} D_1 \left(\frac{R}{\sqrt{2s}}\right)^n = D_1 R^n.\end{aligned}$$

Since L^s interpolates between Σ_s and L on a compact region, L^s satisfies (H1).

We see that (H2) is satisfied because the Lagrangian angle of the initial condition L and the self-expander Σ are bounded, as is that of the rescaled self-expander Σ_s by Lemma 5.2.1 (i) and the maximum principle, since the Lagrangian angle of P is locally constant. When we interpolate between the two, we may consider the formula for the Lagrangian angle of a Lagrangian graph, as seen in [10, pg. 5]. This tells us that a Lagrangian graph in $\mathbb{C}^n$ (over $\mathbb{R}^n$) given by

$(x_1, ..., x_n, u_1(x), ..., u_n(x))$, where $u\colon \mathbb{R}^n \to \mathbb{R}$, $u_i := \frac{\partial u}{\partial x_i}$, has Lagrangian angle

$$\theta = \sum \arctan \lambda_i,$$

where the λ_i's are the eigenvalues of the Hessian of u. Since the eigenvalues of the Hessian of u are some non-linear function of the second derivatives of u, if the C^2 norm of u is small we have that the Lagrangian angle of the graph is close to that of the Lagrangian angle of the plane that u is a graph over. So we can uniformly bound the Lagrangian angle of the graph. Since in our case, the Lagrangian angle of γ_s is given by the sum of arctangents of the eigenvalues of the Hessian of the function w_s, and, as we will show when we prove (H4), the C^2 norm of w_s is small, this means that we can uniformly bound the Lagrangian angle of the graph γ_s, and so the Lagrangian angle of L^s.

On the initial condition, since $\lambda = Jx$, we have that $d\beta_L = \lambda|_L = (Jx)^T$. Therefore, β_L is bounded quadratically, and so is the primitive for the Liouville form of $L^s \backslash B(2s^{1/4})$. On the self-expander, applying the maximum principle to Lemma 5.2.1 (ii), we have β_s (the primitive of $\lambda|_{\Sigma_s}$) is bounded by β_P, and so $|\beta^s(x)| \leq |\beta_P(x)| \leq C|x|^2$ for $|x| < s^{1/4}$. So it remains to check this still holds where we interpolate. We perform a calculation similar to that in the proof of Lemma 5.2.1(ii). We have that, for L^s_t the manifold described by the graph of the one-form tdw_s,

$$\frac{d}{dt}\lambda|_{L^s_t} =: \mathcal{L}_{J\nabla w_s}\lambda|_{L^s_t} = d(J\nabla w_s \lrcorner \lambda|_{L^s_t}) + J\nabla w_s \lrcorner d\lambda|_{L^s_t}.$$

Since $d\lambda = \omega$ and $J\nabla w_s \lrcorner \omega = dw_s$ and possibly adding constant to β^s_t dependent on s and t, we have that

$$\frac{d\beta^s_t}{dt} = -2w_s + \langle x, \nabla w_s \rangle|_{L^s_t},$$

where $d\beta^s_t$ is equal to the restriction of the Liouville form λ to graph of $t\gamma_s$. Integrating, we find that

$$\beta^s = \beta_P - 2w_s + \int_0^1 \langle x, \nabla w_s \rangle|_{L^s_t} \, \mathrm{d}t,$$

where β_P is the primitive for λ on P. Now, w_s is bounded independently of s by $D(1 + |x|^2)$, using (5.6.1) and (5.6.2), as is $\langle x, \nabla w_s \rangle$, using Cauchy-Schwarz and the estimates (5.6.1) and (5.6.2) so we find that β^s is bounded independently of s on the annulus $A(s^{1/4}, 2s^{1/4})$. Therefore, we have that

$$|\theta^s(x)| + |\beta^s(x)| \leq D_2(|x|^2 + 1).$$

and so (H2) is satisfied.

To show that (H3) is satisfied, recall that we define L^s as $L^s \cap B_{r_0\sqrt{2s}} = \Sigma_s \cap B_{r_0\sqrt{2s}}$, $L^s \backslash B_4 = L \backslash B_4$ and we interpolate smoothly between the two, which exactly happens when $s^{1/4} \leq |x| \leq 2s^{1/4}$. Therefore when we rescale by $1/\sqrt{2s}$, we have that $\tilde{L}^s \cap B_{r_0} \equiv \Sigma$. So it remains to check convergence outside this ball.

On the annulus $r_0 \leq |x| \leq 4/\sqrt{2s}$, $\tilde{L}^s$ is identified with the graph of the following one-form

$$\tilde{\gamma}_s(x) = d\left[\varphi(s^{1/4}|x|)v(x) + (1 - \varphi(s^{1/4}|x|))\frac{u(\sqrt{2s}x)}{2s}\right].$$

From this expression, noticing that

$$\frac{u(\sqrt{2s}x)}{2s} \leq C\frac{(2s)^{3/2}x^3}{2s} = C\sqrt{2s}x,$$

we see that as $s \to 0$, $\tilde{\gamma}_s \to dv = \psi$, the one-form whose graph is identified with Σ. This says that, outside B_{r_0}, $\tilde{L}^s \to \Sigma$ as $s \to 0$ smoothly. Therefore we actually have stronger than the required $C^{1,\alpha}_{loc}$ convergence.

Finally, we check that the second fundamental form of $\tilde{L}^s$ is uniformly bounded in s. We have that the second fundamental form of Σ must be bounded, and if A is the second fundamental form of L, rescaling L by $1/\sqrt{2s}$ means that the second fundamental form scales by $\sqrt{2s}$. Since $\sqrt{2s} < 1$, we can uniformly bound both second fundamental forms so that $\tilde{L}^s$, which is a combination of both Σ and $1/\sqrt{2s}L$, has second fundamental form uniformly bounded in s.

To see (H4), first notice that since we can write $L^s \cap A(r_0\sqrt{2s}, 4)$ as a graph over $P \cap A(r_0\sqrt{2s}, 4)$, we have that L^s has the same number of connected components as P in the annulus $A(r_0\sqrt{2s}, 4)$.

We now must estimate γ_s. Firstly, note that we have

$$|\nabla^k(v(x/\sqrt{2s}))| \leq |(2s)^{-k/2}(\nabla^k v)(x/\sqrt{2s})| \leq C(2s)^{-k/2}e^{-b|x|^2/2s}, \qquad (5.6.4)$$

where we have used (5.6.2).

We will need different estimates on $2s\nabla^2 v(x/\sqrt{2s})$ and $2s\nabla^3 v(x/\sqrt{2s})$, which we find as follows.

$$\begin{aligned}|2s\nabla^2 v(x/\sqrt{2s})| &\le Ce^{-b|x|^2/2s} = C\frac{\sqrt{2s}}{|x|}\frac{|x|}{\sqrt{2s}}e^{-b|x|^2/2s}\\ &= C\frac{\sqrt{2s}}{|x|}e^{-\tilde{b}|x|^2/2s}\frac{|x|}{\sqrt{2s}}e^{-\tilde{b}|x|^2/2s} \le \tilde{C}\frac{\sqrt{2s}}{|x|}e^{-\tilde{b}|x|^2/2s},\end{aligned} \tag{5.6.5}$$

where $\tilde{b} = b/2$ and $\tilde{C} = Ce^{-1/2}/\sqrt{b}$, since the function $y \mapsto ye^{-by^2/2}$ is bounded independently of y (by $e^{-1/2}/\sqrt{b}$) on $\mathbb{R}$, and so $\tilde{C}$ is independent of s.

A similar calculation, this time noticing the uniform boundedness of the function $y \mapsto ye^{-by/2}$ for $y > 0$ we can show that

$$|2s\nabla^3 v(x/\sqrt{2s})| \le C\frac{\sqrt{2s}}{|x|^2}e^{-b|x|^2/2s}, \tag{5.6.6}$$

where we make C (which remains independent of s) larger if necessary and b smaller (which does not affect the previous estimates).

We have, using the definition in (5.6.3),

$$\begin{aligned}|\gamma_s| = |\nabla w_s| = |&\varphi'(s^{-1/4}|x|)2s^{3/4}v(x/\sqrt{2s}) + \varphi(s^{-1/4}|x|)2s\nabla[v(x/\sqrt{2s})]\\ &- s^{-1/4}\varphi'(s^{-1/4}|x|)u(x) + (1-\varphi(s^{-1/4}|x|))\nabla u(x)|,\end{aligned}$$

and, using that $s^{3/4} = \sqrt{s}s^{1/4} < \sqrt{s}$ since $s < 1$, (5.6.1) and (5.6.4) imply that

$$\begin{aligned}|\gamma_s| &\le \sqrt{2s}Ce^{-b|x|^2/2s} + \sqrt{2s}Ce^{-b|x|^2/2s} + C|x|^{3-1} + C|x|^2\\ &\le C\left[\sqrt{2s}e^{-b|x|^2/2s} + |x|^2\right],\end{aligned} \tag{5.6.7}$$

where we have made C larger.

Now consider

$$\begin{aligned}|\nabla\gamma_s| = |\nabla^2 w_s| = |&\varphi''(s^{-1/4}|x|)2s^{1/2}v(x/\sqrt{2s}) + \varphi'(s^{-1/4}|x|)4s^{3/4}\nabla[v(x/\sqrt{2s})]\\ &+ \varphi(s^{-1/4}|x|)2s\nabla^2[v(x/\sqrt{2s})] - s^{-1/2}\varphi''(s^{-1/4}|x|)u(x)\\ &- 2s^{-1/4}\varphi'(s^{-1/4}|x|)\nabla u(x) + (1-\varphi(s^{-1/4}|x|))\nabla^2 u(x)|\end{aligned}$$

Using that on the support of φ' and φ'' we have $(s < 1)$ $\sqrt{s} < s^{1/4} \le \sqrt{2}\sqrt{2s}/|x|$, and applying the estimates (5.6.4) and (5.6.5)

$$
\begin{aligned}
|\nabla\gamma_s| &\leq C\left[\left(\frac{\sqrt{2s}}{|x|}+\frac{\sqrt{2s}}{|x|}+\frac{\sqrt{2s}}{|x|}\right)e^{-b|x|^2/2s}+|x|^{3-2}+|x|^{2-1}+|x|\right] \\
&\leq C\left[\frac{\sqrt{2s}}{|x|}e^{-b|x|^2/2s}+|x|\right]. \qquad (5.6.8)
\end{aligned}
$$

Finally, performing a similar computation to those above and combining (5.6.4), (5.6.5) and (5.6.6) we find that

$$
|\nabla^2\gamma_s| \leq C\left[\frac{\sqrt{2s}}{|x|^2}e^{-b|x|^2/2s}+1\right]. \qquad (5.6.9)
$$

Combining (5.6.7), (5.6.8) and (5.6.9), we have that

$$
|\gamma_s|+|x||\nabla\gamma_s|+|x|^2|\nabla^2\gamma_s| \leq D_3\left(|x|^2+\sqrt{2s}e^{-b|x|^2/2s}\right),
$$

where D_3 is a constant independent of s. Therefore (H4) is satisfied.

5.7 Miscellaneous technical results

We collect in this section a few technical results about mean curvature flow in high codimension that were used throughout this chapter. The first is a graphical estimate. Specifically, if the initial manifold can be written locally as a graph with small gradient in some cylinder, then the submanifold remains graphical in a smaller cylinder and we retain control on the gradient. To state this more rigorously we first introduce some notation. The notation and statement of the result are as in [31]. Given any point $x \in \mathbb{R}^{n+k}$ we write $x = (\hat{x}, \tilde{x})$, where $\hat{x}$ is the projection onto $\mathbb{R}^n$ and $\tilde{x}$ is the projection onto $\mathbb{R}^k$. We define the cylinder $C_R(x_0) \subset \mathbb{R}^{n+k}$ by

$$
C_r(x) = \{x \in \mathbb{R}^{n+k} || \hat{x}-\hat{x}_0| < r, |\tilde{x}-\tilde{x}_0| < r\}.
$$

Furthermore, we write $B_r^n(x_0) = \{(\hat{x}, \tilde{x}_0) \in \mathbb{R}^{n+k} || \hat{x}-\hat{x}_0| < r\}$.

Lemma 5.7.1. *Let $(M_t^n)_{0\leq t<T}$ be a smooth mean curvature flow of embedded n-dimensional submanifolds in $\mathbb{R}^{n+k}$ with area ratios bounded by D. Then for any $\eta > 0$, then there exists ε, $\delta > 0$, depending only on n, k, η, D, such that if*

$x_0 \in M_0$ and $M_0 \cap C_1(x_0)$ can be written as $\operatorname{graph}(u)$*, where $u\colon B_1^n(x_0) \to \mathbb{R}^k$ with Lipschitz constant less than ε, then*

$$M_t \cap C_\delta(x_0) \qquad t \in [0, \delta^2) \cap [0, T)$$

is a graph over $B_\delta^n(x_0)$ with Lipschitz constant less than η and height bounded by $\eta\delta$.

The proof can be found in [31]. Next we prove that if an initial manifold M is close to some smooth manifold Σ in $C^{1,\alpha}$, then one gets estimates on the density ratios that are independent of M. See Section 5.3 for the definition of two manifolds being close in $C^{1,\alpha}$.

Lemma 5.7.2. *Let Σ be a smooth manifold with bounded curvature and let $(M_t)_{t\in[0,T)}$ be a solution of mean curvature flow. Fix $\varepsilon_0 > 0$, $\alpha < 1$. There are $\varepsilon = \varepsilon(\Sigma, \varepsilon_0, \alpha) > 0$ and $q_1 = q_1(\Sigma, \varepsilon_0, \alpha) > 0$ such that for every $R \geq 2$, if M_0 is ε-close to Σ in $C^{1,\alpha}(B_R)$ then for every $r^2, t \leq q_1$ and $y \in B_{R-1}$ we have*

$$\Theta_t(y, r) \leq 1 + \varepsilon_0.$$

Proof. This follows immediately from Lemma 5.7.1. Indeed the curvature bound on Σ means that there is a uniform radius r such that for any $x \in \Sigma$, $\Sigma \cap C_r(x)$ is (after maybe rotating) a graph with small gradient over the tangent plane to Σ at x. By requiring that ε is small enough we can therefore ensure that any M_0 which is ε-close to Σ in $C^{1,\alpha}(B_r(x))$ is also a graph with small gradient. It only remains to apply Lemma 5.7.1. □

5.7.1 Local curvature estimates for high codimension graphical mean curvature flow

In [15] Ecker and Huisken proved celebrated curvature estimates for entire graphs moving by mean curvature in codimension one, they then localised these in [16] to prove interior estimates for hypersurfaces moving by mean curvature flow. Analogous results in higher codimension have been proved by Mu-Tao Wang in [58] and [59] respectively. In light of examples of Lawson and Osserman [38] one needs to assume an additional 'K local Lipschitz condition', such a condition is in fact satisfied by any C^1 manifold at small enough scales, so for our purposes there will be no problems applying the estimates. We would like to use the estimates derived in [59] without the time localisation, so we will briefly outline the changes to the proof, though all calculations remain analogous to those used by Wang or Ecker-Huisken. We first introduce the notation used by Wang in [58, 59]. We consider a mean curvature flow $(M_t)_{t\in[0,T)}$ and suppose that locally M_t is given by the graph of some function $u_t\colon U \subset \mathbb{R}^n \to \mathbb{R}^k$ over $\mathbb{R}^n$. As shown by Wang, if we define $*\Omega$ to be the Jacobian of the projection of M_t onto $\mathbb{R}^n$, then one can calculate that

$$*\Omega = \frac{1}{\sqrt{\det(\delta_{ij} + D_i u_t \cdot D_j u_t)}} = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}},$$

where λ_i are the eigenvalues of $\sqrt{(du_t)^T du_t}$. Moreover, for $\varepsilon > 0$ small (depending only on the dimensions n and k), we have that if

$$\det(\delta_{ij} + D_i u_t \cdot D_j u_t) < 1 + \varepsilon,$$

(this is precisely the K local Lipschitz condition of [59] with $K = 1/(1+\varepsilon)$) then $*\Omega$ satisfies the evolution inequality

$$\frac{d}{dt} * \Omega \geq \Delta * \Omega + \frac{1}{2} * \Omega |A|^2.$$

Indeed this follows immediately from calculations in the proof of Theorem B in [58]. To simplify notation slightly we define $\eta := *\Omega$, then one can estimate (following [58])

$$\frac{d}{dt}\eta^p \geq \Delta \eta^p + \left(\frac{p}{2} - p(p-1)n\varepsilon\right)\eta^p |A|^2.$$

We also recall the evolution of the second fundamental form under mean curvature flow yields the differential inequality

$$\frac{d}{dt}|A|^2 \leq \Delta |A|^2 - 2|\nabla |A||^2 + C|A|^4,$$

where C is a dimensional constant. We see that these estimates precisely tell us that we are in the correct setting to apply Lemma 4.1 of [59] with the choices $h = |A|$ and $f = \eta^p$. Following the proof of Lemma 4.1 we find that with φ

defined as

$$\varphi(x) := x/(1 - \kappa x)$$

with $\kappa > 0$ to be determined, we have the following evolution inequality for $g = \varphi(\eta^{-2p})|A|^2$

$$\left(\frac{d}{dt} - \Delta\right) g \leq -2C\kappa g^2 - \frac{2\kappa}{(1-\kappa\eta^{-2p})^2}|\nabla\eta^{-p}|^2 g - 2\varphi\eta^{3p}\nabla\eta^{-p}\cdot\nabla g.$$

We then introduce the cut-off function $\xi := (R^2 - r)^2$ where $R > 0$ is a fixed radius and $r(x,t)$ satisfies

$$\left|\left(\frac{d}{dt} - \Delta\right) r\right| \leq c(n,k) \qquad |\nabla r|^2 \leq c(n,k) r,$$

then following [16] we arrive at

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g\xi \leq -C\kappa\xi g^2 - 2(\varphi\eta^{3p}\nabla\eta^{-p} + \xi^{-1}\nabla\xi)\cdot\nabla(g\xi)\\ + c(n,k)\left(\left(1 + \frac{1}{\kappa\eta^{-2p}}\right) r + R^2\right) g.\end{aligned}$$

It is possible now to also localise in time as in [16], which would get us to the estimates in [59], but for our purposes this is unnecessary, so instead we now suppose that $m(T) := \sup_{0\leq t\leq T}\sup_{\{x\in M_t | r(x,t)\leq R^2\}} g\xi$ is attained at some time $t_0 > 0$, then at a point where $m(T)$ is attained we have

$$C\kappa\xi g^2 \leq c(n,k)\left(1 + \frac{1}{\kappa\eta^{-2p}}\right) R^2 g.$$

Multiplying by $\xi/C\kappa$ we have

$$m(T) \leq \frac{c(n,k)}{C\kappa}\left(1 + \frac{1}{\kappa\eta^{-2p}}\right) R^2.$$

We now choose

$$\kappa := \frac{1}{2}\inf_{\{x\in M_t | r(x,t)\leq R^2\ t\in[0,T]\}} \eta^{2p}.$$

We also fix $\theta \in (0,1)$ and observe that in the set $\{x \in M_t | r(x,t) \leq \theta R^2, t \in [0,T]\}$ we have $\varphi \geq 1$ (since $\eta^{-2p} \geq 1$) and $\xi \geq (1-\theta)^2 R^4$ so

$$|A|^2(1-\theta)^2 R^4 \leq g\xi \leq \frac{c(n,k)}{C\kappa}\left(1 + \frac{1}{\kappa\eta^{-2p}}\right) R^2.$$

Finally as $\eta^{-2p} \geq 1$ and $\kappa \leq 1/2$ we have that $(1 + 1/\kappa\eta^{-2p}) \leq 2/\kappa$, so the estimate

$$|A|^2 \leq \frac{c(n,k)}{\kappa^2 R^2 (1-\theta)^2} = \frac{c(n,k)}{R^2(1-\theta)^2} \sup_{\{x \in M_t | r \leq R^2 \; t \in [0,T]\}} \eta^{-4p},$$

holds in the set $\{x \in M_t | r(x,t) \leq \theta R^2, \; t \in [0,T]\}$. The preceding discussion establishes the following theorem.

Theorem 5.7.3 (High codimension interior estimate)**.** *Let $R > 0$ and suppose that $K_{R^2} := \{(x,t) \in M_t | r(x,t) \leq R^2\}$ is compact and can be written as a graph over some plane for $t \in [0,T]$. Suppose further that if the graph function is denoted by u, that*

$$\det(\delta_{ij} + D_i u \cdot D_j u) < 1 + \varepsilon,$$

where $\varepsilon > 0$ depends only on n and k. Then for any $t \in [0,T]$ and $\theta \in (0,1)$ we have

$$\sup_{K_{\theta R^2}} |A|^2 \leq \max\left\{ \frac{c(n)}{R^2(1-\theta)^2} \sup_{K_{R^2}} \eta^{-4p}, \sup_{\{x \in M_0 | r \leq R^2\}} \frac{|A|^2 \varphi(\eta^{-2p})}{(1-\theta)^2} \right\}. \tag{5.7.1}$$

If we denote by $(\cdot)^T$ projection onto the plane over which M_t is graphical, then it's easy to see that

$$\left(\frac{d}{dt} - \Delta\right) |x^T| = 0$$

for $x = F(p,t)$ some point in M_t. Therefore, defining $r(x,t) := |x^T|^2$ we have

$$\left| \left(\frac{d}{dt} - \Delta\right) r \right| = 2|(\nabla x)^T|^2 \leq c(n,k),$$
$$|\nabla r|^2 = 4|x^T|^2 |(\nabla x)^T|^2 \leq c(n,k) r.$$

With this choice of r we have the following corollary.

Corollary 5.7.4. *Under the assumptions of Theorem 5.7.3, with the particular choice* $r(x,t) = |x^T|^2$ *we have the estimate*

$$\sup_{B_{\theta R}(y_0)\times[0,T]} |A|^2 \leq \min\left\{\frac{c(n,k)}{R^2(1-\theta)^2}\sup_{B_R(y_0)\times[0,T]}\eta^{-4p}, \sup_{\{B_R(y_0)\times\{0\}\}}\frac{|A|^2\varphi(\eta^{-2p})}{(1-\theta)^2}\right\}, \tag{5.7.2}$$

where $B_R(y_0)$ *denotes a ball centred at* y_0 *with radius* R *in the plane.*

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